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THE MATHEMATICS TEACHER

Volume XXXVII Number 1

Edited by William David Reeve

Early American Geometry

By Phillip S. Jones
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Prior to 1820 there were fourteen different books published in America which could be classified under the heading, "Works Dealing with Geometry and Mensuration" as shown in a bibliography recently published by L. C. Karpinski.[1] Of this number, however, only three could be classified with what might be called demonstrative geometry, the rest being works in mensuration and surveying, such as "Hawney's Complete Measurer" (1801), the first book dealing with geometry printed in America.

These first three American demonstrative geometries were: 1803—Robert Simson, "The Elements of Euclid," the second book of geometric nature published here; 1806—John Playfair, "Elements of Geometry"; and 1819—A. M. Legendre, "Elements of Geometry" translated from the French by John Farrar. The first two of these were American reprints of earlier English editions while the last was a translation from the French by a professor at Harvard.

These three texts are important, not only as the first American geometries, but also as long-time favorites of widespread popularity.

In England, it must be remembered, there were dozens of editions of Euclid, many with very little modification from the original or variation from each other. As one of these, Simson's text with its avowed effort to restore the original Euclid won general recognition as a painstaking and scholarly work though perhaps a little partisan in its attributing to Theon and other commentators all the errors in the earlier editions. Altogether, some 30 editions of Simson were published in England, the first in 1756.[2] Playfair's work won its place a little later (the first edition in England in 1795 was followed by nine or more others) as a conservative modification and modernization of Euclid such as would better fit it for the purposes of instruction while still retaining most of the original.

Legendre's "Elements of Geometry," first published in France in 1794 and going to 20 editions in 40 years, presents an interesting twofold contrast when compared with contemporary English and contemporary French geometries. As will be seen later, Legendre when compared with English geometries represented a radical departure from them and from Euclid, omitting, as he did, the postulates and some axioms, reorganizing the order, and,

[1] Karpinski, Louis Charles, *Bibliography of Mathematical Works Printed in America through 1850*, Ann Arbor, University of Michigan Press, 1940, p. 665 ff.

[2] Heath, Sir T. L., *The Thirteen Books of Euclid's Elements*, Vol. I, p. 111.

in some cases, the method of proof. However, in his own France, Legendre's work represented a conservative trend toward Euclid and rigor, away from the loose, intuitive presentations which had become popular there in contrast to the English reverential demand for the original Euclid.[3]

In America, Simson appeared in a total of 11 editions, the last in 1838;[4] Playfair went through 33 editions, the last not appearing until 1873;[5] and Farrar's translation of Legendre saw its tenth and last edition appear in 1841.[6] However, a translation of Legendre by Thomas Carlyle, first published under the name of David Brewster in Edinburgh, in 1822, appeared in America in 1828, "revised and altered . . ." by James Ryan. This translation, with Charles Davies and Professor Van Amringe of Columbia adding their names as editors at later dates, went through 33 editions, the last not appearing until 1890.[7] There is no other work between 1820 and 1840 which can rival these records; in fact, Legendre was used at Yale as late as 1885 where it had competed with Playfair until Euclid was abandoned as a text.[8]

The chief purpose of this article is to indicate the character of early American geometry, some of the points wherein demonstrative geometry in the United States has changed, and whence come some of our present methods and notions.

The discussion which follows is based on a study of the 1806, or second, American edition of Simson, the first (1806) edition and the 1830 American editions of Playfair, the first (1819) and the second (1825) edition of Farrar's Legendre, all in the University of Michigan Library. For purposes of comparison, two popular geometries of a later period, namely, G. A. Wentworth—"The Elements of Geometry," 1878, and Webster Wells—"The Elements of Geometry," 1887, were also surveyed.

The procedure has been to select for discussion and comparison some of those aspects of demonstrative geometry considered most fundamental plus some few isolated topics which seemed of particular intrinsic interest or value. The plan is, then, after a preliminary survey of the general contents of these early texts, to consider, in order, the pedagogical ideas inherent in or advocated by the early works, their attitude toward rigor, the treatment of incommensurables and the limit concept, a few special theorems, and finally a group of miscellaneous topics providing interesting though perhaps minor comparisons and contrasts.

First a general sketch of the content and its organization in these texts should provide a basis for later and more detailed discussion. Simson follows closely the classic presentation of Euclid. His text includes, in addition to the first six, the eleventh and twelfth Books of Euclid, Euclid's Data and a treatise on plane and spherical trigonometry.

Although Playfair states his object as "not Simson's, to restore the original Euclid," but rather to put demonstrations into the form that "may at present render them the most useful," nevertheless, the first six books of Playfair are essentially the same as Simson's, the major alteration here being the use of algebraic symbols and operations in Book II on geometric algebra and in Book V on proportion. A greater difference is found when, in the place of Simson's Books XI and XII, Playfair adds a Supplement in three parts. The first part discusses the quadrature of the circle and computes a value for π. The second and third parts of the Supplement follow the solid geometry of Books XI and XII of Simson in order and statement with the omission of some of the more complicated theorems and a somewhat closer approach to a modern treat-

[3] Simons, Lao G., *Fabre and Mathematics and Other Essays*, Scripta Mathematica Library No. 4, p. 52.
[4] Karpinski, *loc. cit.*, p. 149.
[5] Karpinski, *loc. cit.*, p. 163.
[6] Karpinski, *loc. cit.*, p. 228.
[7] Karpinski, *loc. cit.*, p. 11.
[8] Simons, *loc. cit.*, p. 59.

ment of mensuration and the theory of limits.

However, it is when we survey Farrar's Legendre that we find the real variation from classical order. Farrar divided Legendre's text into two "Parts," plane and solid geometry, and then divided each Part into four "Sections" rather than "Books." In plane geometry, Section 1 deals with lines, parallels, perpendiculars, isosceles and congruent triangles. The order of propositions and the manner of the proofs differ, but the content is much the same as Book I of Simson and Playfair with the omission of the constructions and the Pythagorean theorem. Legendre then goes directly to the circle and the measurement of angles in a circle in Section 2, omitting the geometric algebra of Euclid's Book II as he later omits entirely the traditional juggling of proportions by composition, inversion, etc., of Book V. At the end of this section on the circle Legendre places the constructions for parallels, perpendiculars, triangles, and parallelograms omitted from Section I. Section 3 on the "Proportions of Figures" covers the topic of Simson's and Playfair's Book VI in a simpler fashion, but it goes beyond them to lay the foundation for Legendre's Section 4 by an early discussion of incommensurables. Section 3, in addition to giving constructions for proportional lines and figures, derives and states in symbolic form formulae for the areas of rectilinear figures and discusses similar polygons, intersecting chords and secants, and the Pythagorean theorem. Interestingly, however, Legendre harks back to Euclidean geometric algebra to give geometric proofs of the expansions of $(a+b)^2$, $(a-b)^2$, $(a+b)(a-b)$, appending, however, an additional algebraic formulation of each. The actual measurement of the circle is discussed in Section 4 along with the regular polygons. The discussion of the latter is much less detailed than that of our other two authors' Book IV on inscribed and circumscribed polygons.

The above outline of Legendre's Geometry is very nearly the pattern for the plane geometry of the latter half of the century, for Wells and Wentworth vary from it in general organization merely by breaking Section 3 up into two separate parts, the first on proportional lines and polygons, the second on the areas of polygons.

Having surveyed the general organization of these early texts, we next note the thought given and concessions made to the problems of teaching as such. Let us listen to these authors' statements as to the purpose and value of the study of geometry, still a moot pedagogical question: Legendre states that the value of the "method of the ancients" is to "accustom the student to great strictness in reasoning" and that it is also a discipline which may be of value in mathematical research; Simson cites geometry as a science by which the investigation and discovery of useful truths, "at least in mathematical learning," is promoted and then asserts that it is also "of great use in the arts of peace and war," thus setting forth a modern utilitarian appeal at an early date. The 1806 edition of Playfair contains no preface, but the 1830 edition, copied from a different London edition, includes a preface dated 1813. In this Playfair replies to the objection that Euclid proves obvious theorems by stating that "the end of Mathematical Demonstration is not only to prove the truth of a certain proposition, but to show its necessary connection with other propositions and its dependence on them." He further states that the mind of those beginning "the study of the art of reasoning" cannot be better employed than "in analyzing judgments which appear simple but which are complex," thus implying that geometry is, or at least is closely related to, the "art of reasoning."

Legendre probably gives the most consideration to the pedagogical problems involved in the conflict between teaching geometry for training in logic and modifying the logic of geometry to make it more teachable. He apologizes at the outset for

the sacrifice of "exactness" in the interest of making material not too difficult.

The modernity of Legendre is found again in his contention that "It is advantageous to carry on the study of algebra and geometry together and to intermix them as much as possible." This doctrine he applies by introducing the use of algebraic symbols both literal and for operations, grouping, powers, and roots, and also by giving algebraic proofs solely for numerous theorems. Playfair, too, uses symbols, though less extensively, gives alternative algebraic proofs for some geometric theorems, and treats proportion entirely algebraically explaining that the conciseness of algebraic language brings the steps in reasoning near to each other and remedies the prolixness and obscurity of Book V. Simson, however, compartmentalizes his mathematics to such an extent that he does not even use in his geometry the symbols for operations and grouping that appeared in the trigonometric section of the same book, nor does he even hint that Propositions 12 and 13 of Book II are actually the same as the cosine law found in the earlier section.

Let us turn now to the actual treatment of the problem of rigor in so far as revealed in the definitions, axioms, and postulates found in these texts and particularly in their treatment of the parallel postulate.

In none of the texts surveyed is there any hint as to the possibility of the existence of such things as undefined terms, nor any discussion of the role of definitions in a logical system. However, Legendre does hint at the importance of definitions when he blames all the past difficulties over the parallel postulate on the imperfect definition of a straight line, and he comes closest to emphasizing properly the role of definitions when he defines a straight line as the shortest distance between two points, adds that there can be only one line between them, and then states that "upon this principle considered both as a definition and as an axiom I have endeavored to establish the entire edifice (of geometry)." So much for that.

Neither Simson nor Playfair defines axiom or postulate nor in any way discusses their roles. Legendre gives the time-honored definition of an axiom as a self-evident truth which definition appears later in Wells and Wentworth (the latter calls it a truth "admitted without demonstration").

Perhaps some comparison of the axioms themselves is of interest. Simson gives twelve, the first seven on equalities and inequalities; the eighth, that magnitudes which coincide are equal; ninth, the whole is greater than its part; the tenth, two straight lines cannot enclose a space; eleventh, all right angles are equal, and finally the twelfth, the parallel axiom. Playfair omits the tenth, explaining in an informal discussion that this was implied in his definition of a straight line. This is an example of that which in later writers becomes a widespread habit, namely, of setting forth useful principles in "discussions" as neither definitions, axioms, nor theorems. Playfair also gives a different parallel axiom, and in his Supplement adds two axioms, that a straight line is the shortest distance between two points (Legendre's "defintion" of a straight line) and that if two figures have the same straight line for a base, that which is contained in the other has the least perimeter if its boundary is never convex toward the base.

Legendre gives only five axioms: (1) things equal to the same thing are equal to each other, (2) the whole is greater than its part, (3) the whole equals the sum of its parts, (4) only one straight line can be drawn between two points, (5) magnitudes which coincide are equal. Thus Legendre gives two new axioms, (3) and (4), the first of which appears later in Wentworth, and both of which appear in Wells and many modern texts.

The point where all three of the early texts admit difficulties is the parallel axiom. Both Playfair and Simson place it in quotation marks and write notes on

it. Simson contends that it is not an axiom "as indeed it is not self evident (note the implied definition of an axiom) but may be demonstrated." He then proceeds to give a demonstration based on two new definitions and a new axiom. He defines the distance from a point to a line, and he defines that one line "goes nearer to or farther from" another when the distances of the points of the first from the other straight line become less or greater than they were. To these definitions he then adds the axiom that a straight line cannot come nearer to another and then go farther from it before it cuts it. With this start he then, in the interest of rigor, proceeds to prove five theorems, the last of which is his parallel "axiom."

As noted earlier, Playfair gives his own parallel axiom but even so feels it is questionable. He is so sceptical that in his notes on proposition 29 of Book I, the first one where the axiom is used, he gives a second proof of this proposition, not using the axiom. This alternative proof is based on a new definition of an angle, that, if while one extremity of a line remains fixed at A, the line turns about that point from AB to AC, it is said to describe the angle BAC contained by the line AB and AC. This definition is of interest as the only mention in these three early works of the idea of an angle as formed by a rotation.

Legendre, as noted earlier, gives no parallel axiom at all. However, he inserts as Paragraph 58 a lemma that if two lines are perpendicular and a third line makes an acute angle with one it will meet the other. With this admitted, he then proves four theorems, the last of which is: if two lines are parallel, the alternate interior angles are equal. However, immediately following the above lemma, Legendre states that his demonstration of it is not rigorous since it is based on measuring distances on the diagram and, in the notes, he calls it a special case of Euclid's parallel postulate which, he says, has not yet been demonstrated in a manner entirely geometric. He adds that "more abstract analysis" makes a rigorous proof simple, and proceeds to attempt one using functional notation and concepts.

A fundamental difficulty which since it goes back to Pythagorean days, is even older than that with parallelism, is the problem of the incommensurable. This problem is inevitably tied with the study of proportion and its application to lengths, areas, volumes, and angular measure, the value of π, and the modern theory of limits.

The modern limit concept, though extended discussion of it is later found in Wentworth and Wells, is not mentioned in any of the three early texts. However, the method of exhaustion as used by Simson, and with modifications by Playfair and Legendre, contains a real approach to modern theory.

Simson and Playfair in Book V give the Euclidian definition of proportion, the earliest device to avoid discussion of incommensurables. Clothed in modern symbolism this states that: given four quantities a, b, c, d and multipliers m and n, then one of the three relationships shown must be true for each of the pairs of products

$$ma \gtreqless nb \qquad mc \gtreqless nd$$

If for all possible choices of m and n the same relationship holds for both pairs, then $a/b = c/d$.

Though this definition is modern in spirit, being equivalent to a definition of irrational numbers by means of a Dedekind cut, nevertheless, it made it possible for Simson to avoid any explicit discussion of incommensurables. In fact, true to his Euclidean ideal, he not only avoids discussion of incommensurables, but he never even suggests that this theory of proportion could be phrased in numerical terms, or that it in any way concerns number. This, of course, means that nowhere in his text does he discuss mensuration, nor give any mensuration formulae, although the background of such problems is provided in theorems on the proportions between

two similar figures and between the lengths, or squares or cubes "on" the lengths of certain lines. The contrast and progress of the ideas in the early texts are interesting. Playfair gives these same definitions and proofs but notes that, though difficult, the Euclidean doctrine of proportion can be reconciled with algebraic and numerical concepts of proportion. Playfair even defines the "area" of a figure in his Supplement and gives formulae for the area and circumference of a circle whereas Simson's closest approach to mensuration is in such theorems as XII, 2, where he proves that circles are to one another as the squares of their diameters. Playfair's discussion of the relation between numbers and magnitudes in a scholium in the Supplement is, however, definitely misleading. This scholium states that to have numbers proportional to magnitudes, let one magnitude be divided into m parts, H; then let H be found in others, n, r, s, times, then m, n, r, s are proportional to the magnitudes. This statement blithely ignores the possibility of incommensurables nor does he elsewhere hint at such a thing except by implication when, after computing a value for π, he adds that neither by this nor by any method "yet known" can the circumference or area of a circle be exactly determined.

In line with his policy of intermixing algebra and geometry, Legendre treats proportion by purely algebraic and numerical methods, using them as tools with which the reader is presumed to be familiar. (Farrar added a preface from Lacroix' Algebra to supply any deficiency of American readers along this line.) However, the Euclidean concept of proportion was mentioned as late as 1878 when Wentworth is careful to demonstrate as did Playfair that the Euclidean definition is equivalent to the algebraic.

Since this arithmetization of geometry played an important part in clarifying the concept of incommensurables as well as in the rigorous foundations of mathematical analysis as a whole, it is interesting that Legendre emphasizes that in geometry the product of two lines, or the rectangle upon them, means the product of two numbers when carried over into arithmetic. He even generalizes this concept algebraically in several places such as in paragraphs 58 and 59 which first prove that the rectangle contained by the sum and difference of two lines is equal to the difference of their squares, and then adds that this is equivalent to the algebraic formula $(a+b)(a-b) = a^2-b^2$.

Legendre explicitly recognizes and deals with incommensurables throughout. The first and a typical instance of this is found in the theorem: In the same or equal circles inscribed angles are proportional to the subtended arcs. This theorem is proved in two parts, giving a numerical example for the commensurable case, and using a *reductio ad absurdum* for the incommensurable case. From these theorems Legendre derived the fact that inscribed angles are measured by one half the subtended arc, hence, that all angles in a semicircle are right angles; and that the angle between a tangent and a chord is measured by one half the intercepted arc. This somewhat numerical and mensurational concept of angular measure is not found at all in the other two texts. The theorems there are much more limited in form, dealing with relations between pairs of particular angles in the same or equal circles rather than the mensuration of a single angle.

At the end of his discussion of angle measure, Legendre proves two related theorems—one for lines and one for angles—giving a preliminary discussion of common measures and incommensurables and looking toward the area formulae to be developed in Section 3. The first of these theorems explains a construction for finding the *numerical* ratio of two given lines, providing they have a common measure. This is done by laying the smaller off on the larger, the remainder off on the first, etc., until a common measure is reached

when no remainder is left. Legendre himself explains that this is the same method as used in arithmetic for finding the common divisor of two numbers and that the method would serve for finding the common measure of two angles if applied to their subtended arcs. He then specifically states that it is possible that we shall have no common measure in which case the two quantities are said to be "incommensurable."

By again using a *reductio ad absurdum* argument Legendre proves in Section 3 that rectangles with the same altitude are proportional to their bases and then in a scholium explains that if a square of unit side be taken as a unit of surface, the area of a rectangle is the product of the base and the altitude. He then goes on to express symbolically mensuration formulae for the area of the triangle and trapezoid.

Legendre, in keeping with this ready recognition of incommensurables and his arithmetization of geometry, proves the side and diagonal of a square to be incommensurable as a corollary of the Pythagorean theorem and later computes $\sqrt{2}$ by the continued fraction, $1+1/2+1/2+1/2+\cdots$, derived geometrically.

In Section 4, a *reductio ad absurdum* again serves to prove that the circumference of a circle is proportional to the radius, and the area, to the square of the radius, and the area equal to the circumference times one half the radius. At this point, Legendre lets π be the circumference of a circle with a unit diameter and from the proportion $1:d::\pi:C$ he can write $C=\pi d$. The area is then found to be πr^2 by multiplying both sides of this equality by $\frac{1}{2}r$ in accord with the theorem stated previously that one half the radius times the circumference is the area. All this is followed by two different methods of computing approximations to π, one by finding value of the areas of polygons of $2n$ sides inscribed in and circumscribed about a given circle. The second method proceeds by calculating the radii of circles inscribed in and circumscribed about a given polygon. He carries the computation decimally to 3.1415926, and also gives the Archimedean value of π as between 3–10/71 and 3–10/70, suggesting 22/7 as a good approximation. In conclusion Legendre states that the exact value of the ratio of the circumference to the radius is now ranked as an idle question (sic!) engaging the attention of beginners only since the approximation can be carried as far as is desired and hence there would be no real advantage in knowing the exact value.

Playfair begins his discussion of the mensuration of the circle in Book I of the "Supplement" by giving the two theorems common to Simson and Euclid as bases for the method of exhaustion. The first is the real heart of the method and indicates an approach to the theory of limits; it reads: "If from the greater of two unequal magnitudes there be taken more than its half, from the remainder more than its half, etc., there shall at length remain a magnitude less than the least of the proposed magnitudes." The method of exhaustion comes in, of course, in making the jump from the ratios of inscribed polygons to the ratios of the circles themselves. The procedure is to show by a *reductio ad absurdum* that the ratio of the squares of the diameters cannot be the ratio of the larger circle to a space greater than (or less than) the smaller circle. This theorem is Simson's last word on the mensuration of the circle.

Playfair gives the first two theorems essentially as in Euclid and Simson. By permitting the assumption of certain lines and areas without having given a construction for them, he avoids from then on repetition of the "exhausting" part of Euclid's method. Having so far abandoned the classical, Playfair goes on to give formulae for the area and circumference, to discuss numerical measures, and to derive values for π much as did Legendre.

In general, the same attitudes and methods follow through the discussion of solid geometry in the respective texts. It is interesting that in finding the area of

a sphere Legendre uses the theorem common today that the area of the surface formed by a regular polygon rotated about an axis through its center equals the projection of its semicircumference on the

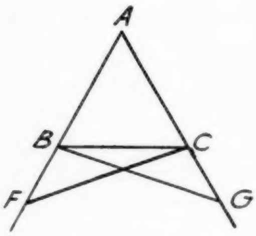

Fig. 1

axis times the circumference of the inscribed circle.

Two other Euclidean theorems have always drawn the interest of the author. Playfair and Simson treat the "pons asinorum," the theorem that the base angles of an isosceles triangle are equal, using the classical diagram (Fig. 1) and proof, namely, by taking $AF=AG$, proving first $\triangle AFC \cong \triangle AGB$ and then $\triangle FBC \cong \triangle GCB$, thence $\angle ABC = \angle ACB$ by subtraction of equal angles from equals. Legendre, in more modern fashion (Fig. 2), merely draws a line from the vertex to the midpoint of the base, thus, proves $\triangle ABM \cong \triangle AMC$ and hence $\angle ABC = \angle ACB$.

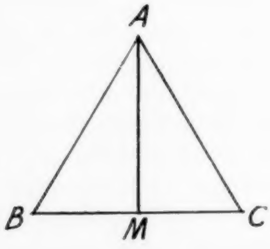

Fig. 2

An interesting contrast exists with regard to the construction of tangents to a circle, also. All three of our early texts construct the tangent at a point on the circumference by erecting the perpendicular to the radius drawn to the point of contact, but Playfair and Simson follow the Euclidean procedure in drawing the tangent from an external point A.

To do this (Fig. 3), they first join A to O, the center of the given circle, then erect DF perpendicular to AO at D; thirdly,

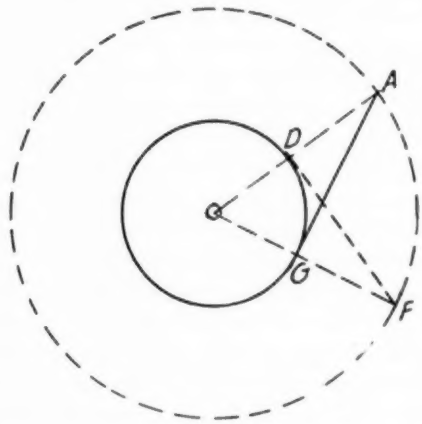

Fig. 3

draw the circle from O as a center through A, then join O to F, $-AG$ is the required tangent. Legendre's construction (Fig. 4) is simpler, emphasizes the fact that there are two such tangents, a fact not mentioned nor shown in the other texts, and further, is the modern method. The construction is as follows: with the midpoint

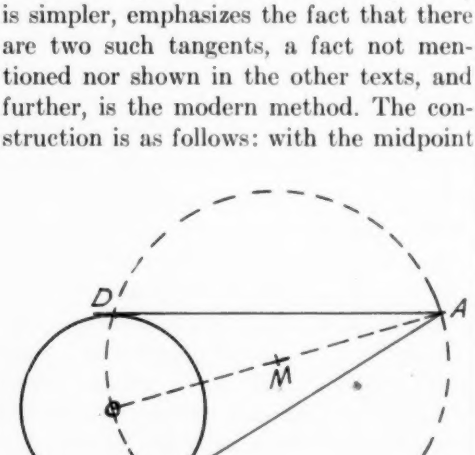

Fig. 4

of AO as a center draw the circle through O. Join the points of intersection D and G to A to form the two tangents.

Other interesting items met in Legendre, Wells, Wentworth and most modern texts

but not found in Playfair or Simson are: a section on isoperimetric polygons, drawings for construction of "pasteboard" models of the regular polyhedra, and the terms "tangent," "chord," "arc," "area," "volume." The use of "area" and "volume" carry significance beyong the purely philological as again indicating the increased use of algebra and the arithmetization of geometry with its accompanying recognition of incommensurables and use of symbols and formulae. Playfair does use "area" in his Supplement, the portion where he departs most from Simson and Euclid; he also uses "arch" for Simson's "circumference" and Legendre's "arc."

This survey of early American geometry would not be complete without noting a few modern features which are lacking in the earlier works. Exercises are entirely absent; there is no mention of the locus concept, nor of the related function concept either from the viewpoint of related groups of theorems or from the viewpoint of related variables. The possible exceptions to this latter statement are to be found in one point where Legendre specifically calls attention to the fact that the theorems on the ratio of the segments of intersecting chords, intersecting secants, and a tangent and a secant are really special cases of the same theorem, in his grouping of the Pythagorean theorem and the two theorems amounting to the cosine law together as the "fundamental theorems of geometry," and in the possible interpretation of the use of algebraic formulae as a step toward the idea of related variables and functionality. The only point where any approach is made to the idea of symmetry in these early texts is found when Legendre calls attention to the fact that Simson's theorems on spherical triangles are not complete because of the use of superposition without any recognition of the possibility of exceptions in the form of symmetric spherical triangles, which latter Legendre thereupon defines and discusses.

In summary, it might be observed that a ranking of the early texts as to date, modernity and popularity would in each case give the same order from earliest to most recent, least modern to most modern, least to most popular (in America), namely: Simson, Playfair, Legendre, and it doesn't seem that one would be guilty of a post hoc, propter hoc fallacy to conclude that Legendre was the forebear of modern American demonstrative geometry.[9] Through these texts one can observe a steady trend toward the breaking down of the classical Euclidean presentation and of the compartmentalization of mathematics by the introduction of symbols, formulae, and algebraic methods. At the same time, an increased consideration for the problems and aims of teaching manifests itself in the attempts by use of symbols, omission of some theorems and the simplification of the statement and proof of others, to make the study better suited to those who were to study it. This added attention to pedagogy leads to the contradiction found in emphasis on the logical and disciplinary values of geometry while the true rigor of geometry is being rather insidiously attacked by increasing, without adequate recognition or discussion, the number of postulates. One gain in favor of rigor is the recognition and improved treatment of incommensurables; this, however, is also of pedagogical value, not only because the accompanying arithmetization and emphasis on mensuration makes the subject seem both more concrete and more real, but also because the change was accompanied by the elimination of the abstruse Euclidean theory of proportion.

It is to be hoped that this study will also show those who bemoan the lack of change in geometric instruction that there have been changes, that much now considered new and good is not new, and hence suggests that possibly not all that is old is necessarily bad.

[9] Simons, *loc. cit.*, p. 44 ff., gives a more general discussion of "The Influence of French Mathematicians . . ." as do also Smith, D. E., and Ginsburgh, J., in *A History of Mathematics in America Before 1900*, 1934, p. 77 ff.

New Horizons Through Integrated Mathematics

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It is impossible in the space devoted to this paper to cover completely the topic of integration. Nor is it the purpose of this report to conclude the topic here. However, from the ideas presented, be they old or new, there should come some thought which will bring profitable results in the mathematics class room.

The subject of integration is an old one. Various methods of integrating subject matter have been used for many years. Scores of definitions have come from the pens of educational leaders. Hopkins in his book "Integration—Its Meaning and Application" states that "Pedagogically, integration is used to describe a teaching process which relates varieties of subject-matter to units of study or to problem-solving situations." He further states that the first law of integration is that of a "unitary whole."

From these definitions it is seen that the entire field of mathematics must be looked upon in the light of end results. The seventh grade teacher must have knowledge of the work covered in the twelfth grade to ever be alert in teaching those fundamental laws and processes which will contribute to the finished product. Too often teachers feel that their subject is an entity in itself believing that their job is completed when the pupil is passed into the next grade or subject. Then, too, the teachers of high school pupils must have in mind the work covered in the preceding years, and avoid the often repeated excuse that "pupils are not properly prepared when they come to high school work." There needs to be more careful coordination of work among teachers of all grades. This to the end that the finished product will be what society demands from its high school graduates.

During the first six years of school the fundamental processes are being established in the minds of the pupil. These processes are vital to the success of the high school pupil. There can be integration in the work below the seventh grade but the main purpose of the first six years is to supply the tools needed for the more advanced work to follow.

Beginning with the seventh grade integration should take on an ever-increasing amount of importance so that the twelfth grade will be the finishing part of the mathematics program. This may be thought of as requiring a vertical type of inter-relationship among the fields. From a very elementary type of geometry and trigonometry in the seventh grade to the more advanced type in the twelfth, the pupil learns through steps applicable to his mental development. He does not think of grade placement but of the field as a "unitary whole."

Examination of the details shows many interesting possibilities. These details will be limited to the college preparatory field because integration to this moment has been developed, largely, in the general mathematics curriculum. A fine job has been done in this field, but there still remains the need for integration in the college preparatory group. From the seventh grade to the twelfth there may extend three strands of material for the college preparatory student. The first strand would include that work ordinarily covered by the traditional grade material now being taught. The fundamentals cannot be omitted nor neglected. The second strand would cover experimental geometry and intuitive geometry (informal) leading to the work covered in plane geometry (formal). The third strand would develop the processes of analysis culminated with analytical geometry and the calculus in the twelfth grade. This type of program will prepare the student for modern college entrance examinations as well as give him a better foundation for his college mathematics.

The seventh grade pupil can develop

the fundamental concepts of relational thinking such as the principle that the sum of the angles of a triangle equals 180°. This is done by constructing a triangle, cutting off the corners and placing them in such a manner that their sides form a straight line. The pupil has learned previously that a straight angle contains 180° by the use of the protractor. He has constructed angles from zero to 180°.

After several experiments have been concluded in various types of triangles and the results found to be 180° in each case, the question should arise, "Is this true of all triangles?" The pupil then tries to find a case where the rule does not apply. He soon finds that the rule seems to apply always, but the proof must not stop here. The idea of going from the known to the unknown must be developed early. The pupil knows that a straight angle contains 180°. Therefore, when it is proved that the angles of a triangle together form a straight angle and that a straight angle equals 180° the conclusion may be drawn that the angles of a triangle equal 180°. He has arrived at a definite method of proof.

Here is another example; the old one about vertical angles being equal. The teacher draws two intersecting lines on the board and the pupils follow suit. Then with scissors the pupils cut out the angles and discover that they seem to be equal. This conclusion is checked by folding the paper in such a manner that the desired angles are found to be equal. But a third check is by the use of the protractor. From these three methods the pupil has established the desired conclusion. He may ask if such a method constitutes a valid proof. For this grade such a means will have to be regarded as sufficient.

In the eighth grade the topic of vertical angles is presented again with actual proof concluded after the pupil has had simple equations. He recognizes that there are four straight angles made by two intersecting lines. With the use of the protractor he finds the size of the various angles and concludes that vertical angles are equal. If he lets x equal one angle he knows that two angles will be $180 - x$ degrees. And by subtracting the second angle from 180° he discovers that the fourth angle is x or equal to the first angle. Thus, he recognizes that there are two pairs of equal vertical angles, and since x can equal any desired size the pupil can conclude that this is a proof.

A new topic in the eighth grade may be that of parallel lines and their angles. The teacher draws a line on the board and a transversal cutting the line. Then through a point on the transversal a second line is drawn obliquely but is constantly changed until the pupils recognize that it will never meet the original line. The pupil will then form his own definition of parallel lines. He may say that they are lines which are always the same distance apart. This could serve as a basic definition but the pupil will find by using the protractor that the corresponding angles are equal. The term "corresponding angles" may not come from the class, but some such definition will which may be changed slightly to mean the same thing. From this fundamental concept of corresponding angles the pupil can prove that the alternate angles are equal and he may find the sum of the angles on the same side of the transversal.

Now back to the sum of the angles of a triangle and another method of proof. Through the vertex of the triangle draw a line parallel to the base, and by application of alternate angles of parallel lines the sum of the angles will be found to equal a straight angle or 180°.

In the eighth grade, also, the first three cases of congruent triangles can be developed by the laboratory method. The protractor is used in the construction of the equal angles and the ruler to measure the sides. No proof is given at this point other than holding the triangles up to the light and concluding that they coincide in every respect. From the three cases of congruent triangles many constructions

may be made and proved. Notice that constructions are to be proved, thus developing the idea of the necessity for proof in mathematical thinking. The protractor and the ruler are used to construct the perpendicular bisector of a line, to copy an angle equal to a given angle, and possibly to bisect an angle. Proving that an isosceles triangle has two equal angles may be developed at this time. Space does not permit going further with examples, but there are almost unlimited numbers of examples even for the eighth grade student.

The work of the ninth grade should include many fundamental definitions such as: line segment, angles, size of an angle, perpendicular, adjacent angles, types of angles, supplementary and complementary angles, types of polygons, kinds of triangles, use of the transversal, parallel lines, quadrilaterals, and parts of a circle. Here, also, the pupil uses the compasses for the first time in fundamental constructions, and he proves that his constructions are correct by congruent triangles learned in former work. He constructs triangles given two sides and the included angle, or two angles and a side, or three sides. In other words, he develops the desire for actual proof. What a wonderful aid to the teacher of geometry! By the ruler and compasses the pupil constructs various triangles under given conditions; he bisects an angle; he constructs a perpendicular bisector to a line; he draws a perpendicular to a point on the line or from a given outside point to a line; and most important of all he proves that his constructions are correct. The geometry of the ninth grade may conclude with a proof of the Pythagorean theorem. This is developed from former work with similar triangles. While dealing with ratio the pupil learned that two triangles are similar if they have their corresponding angles equal, and that corresponding sides of similar triangles are in ratio. By drawing a perpendicular from the right angle of the triangle to the hypotenuse the pupil will discover the similar triangles, and by the laws of proportion the Pythagorean theorem will take shape.

In the tenth grade the regular work of plane geometry is carried on with constant applications to algebra. The work with circles including intersecting chords, secants and tangents give splendid examples for solving quadratic equations. Similar triangles, right triangles, problems dealing with areas, and work with quadrilaterals, offer opportunity for review of algebraic principles. Most geometry texts give splendid applications of integrated materials in geometry and algebra.

It is not necessary to outline the work of the eleventh and twelfth grades with reference to plane geometry. Most texts in the field bring in applications of the geometry in the work of intermediate algebra. However, as problems arise which need geometric proof, time should be taken to review the proof. An example is found in the work with mean proportionals found in tangents and secants. If such a policy were followed the student would review his geometry constantly.

In the eleventh grade, also, could be placed a thorough course in logarithms which would eliminate the necessity for teaching the topic in the twelfth grade except for a brief review. Introducing the use of the sine, the cosine, the tangent, and the cotangent could take place in the geometry of the eleventh grade. Then the combination of logarithms and the functions used in the solution of triangles would save much valuable time needed in the twelfth year. Pupils in the eleventh grade can develop the right concepts and discover the correct uses for the work in logarithms. As now introduced, the pupil receives only a small sampling of trigonometry while he is capable of more thorough mastery. Why not carry logarithms to a better conclusion?

The third strand of mathematics to be carried through from the seventh to the twelfth grade is that of analysis. Many pages could be written on its development. However, only a few examples will be presented in this paper.

In the seventh grade the pupil is introduced to the use of formulas as a short method of stating a complicated rule. Several formulas may be placed before him, such as the formula for the area of a triangle. By analysis he will discover the rule. The seventh grade pupil should secure some experience in graphs, the meaning of direction, and learn how to locate points in the various quadrants. After two points are located a line is drawn joining the points, and the pupil is beginning to graph a straight line. Of course no mention is made of the equation used. However, he can develop the meaning and the use of the "steepness" of the line and discover how it is determined. Through this work the pupil learns how to use graph paper and in a small way the location of points on the graph.

In the eighth grade the pupil continues with work in graphs, constructing temperature graphs, line and bar graphs, and even circle graphs. In some classes a few simple equations could be plotted. But by the end of the eighth grade the concepts of locating points, drawing lines, and finding the slope of a line from the graph have been established.

The work in graphs takes on a greater meaning for the pupil in elementary algebra. Here he plots several examples of the type $y=mx$; he finds the slope of such lines, and sees that the line always passes through the origin. He should discover by experimentation that the slope is m, and that the line crosses the y-axis at zero. The second type of line to be examined is represented by the equation $y=mx+b$, and after many examples the pupil will conclude that the slope is m and the y-intercept is b. He will discover that changing b does not change the slope of the line and that the lines are parallel. He can prove that the lines are parallel by the application of corresponding angles which he has had previously. The pupil will see, also, that changing m does not change the y-intercept but does change the slope. The meaning of axis, origin, and intercept must be developed and explained as carefully as possible during all of this work.

During this work the pupil develops the meaning of positive and negative slope concluding with the formation of equations given the slope and the intercept or given two points. This is accomplished by locating the points on the graph and developing the equation from observation. It is also possible to find the sides of the triangle or the length of the line connecting the two points by the use of the Pythagorean theorem developed in the eighth grade.

In the tenth grade analysis is developed through the treatment of locus. Here a discussion of what it is and what are the conditions needed for a locus are introduced and explained. The idea of a point moving in a path under certain conditions leads to the graph of a straight line and a quick review of slope and intercept. The straight line is followed by work with the *circle*, the *ellipse*, the *hyperbola*, and the *parabola*, with the resulting type equations needed for each. By use of the Pythagorean theorem these equations may be proved. The *cycloid* and problems in *linkages* may be used to conclude the work of the tenth year provided the class is able to comprehend such relationships.

Functional relationships as shown by graphs are developed in the eleventh year. The equation $f(x)=ax+b$ is placed on the graph with variations in a or b or both. The slope of the line is noted as well as the intercept. From simple equations the pupil proceeds to the equation of the parabola where he notices several important principles. First, when the parabola does not cross the x-axis the roots are imaginary. Second, if the curve is tangent to the x-axis there are two equal roots. Third, if the curve crosses the x-axis in two places there are two distinct roots. He will notice that a line has a positive slope when y varies directly with x, but a negative slope when y varies inversely with x. Understanding $f(x)$ is very important in this grade.

The concluding course in analysis is in the twelfth year where applications are

made to previously learned facts with new materials introduced which apply to analytical geometry or to the calculus. Of course the twelfth year will include work in solid geometry. This should be as brief as possible with emphasis on special concepts, problems and applications of the common formulas, and a thorough study of the sphere. Many formal proofs are omitted except as they apply to formulas used. The sphere is studied carefully because it has many applications to advanced mathematics and to work in navigation. Such a course in solid geometry should consume about one-third the complete course for the twelfth year.

The remaining part of the twelfth year could be devoted to the trigonometry and analysis needed for a thorough foundation for college mathematics. The usual course in trigonometry should be given with much additional materials in such topics as: complex numbers, their solutions and graphs; inverse trigonometric functions; determinants with two or more equations; the exact rate of change or the derivative; graphing a cubic equation and finding the maximum and the minimum or critical points; theory of limits and the slope of the curve; differentiation and integration; theory of equations; and possibly some work in permutations and combinations. The work of this grade would necessarily be elementary and none of the topics could be covered completely. However, the materials would be determined by the ability of the class to carry on the work in a satisfactory manner.

Now for a retrospective glance of the entire program. Every step in the process has been directed toward the end result or a careful preparation for college mathematics. That is the purpose of the college preparatory curriculum, and many college teachers will state that the present program is not always satisfactory. In the program outlined in this paper some new materials have been introduced or new methods of applications found. This necessitates the elimination of some materials now being taught in the various grades. Here are a few suggestions but there are many others. Less time could be spent on review in the seventh and eighth grades, with additional units in the work suggested in this paper. Possibly the pupils have developed a feeling that principles not learned in the grades will be reviewed and established in the higher grades. However, an efficient mathematics program includes a place for each topic and the alert teacher sees that her students receive the necessary knowledge intended for that particular grade.

In the ninth grade a large amount of time is spent in reviewing equations and checking concepts which should have been developed in the eighth grade. Also, much time is wasted in teaching four cases of factoring when one case, the fourth, would bring the desired results. Much time could be saved in this manner and placed aside for the work in geometry and analysis. In the eleventh year a great amount of time is spent in reviewing the work covered in elementary algebra. More careful integration would eliminate the need for so much review.

Throughout the entire mathematics program there should be close cooperation among the teachers in the department. Such a policy would develop a school system in which each teacher knows what to teach and other teachers know what has been taught.

There are "New Horizons Through Integrated Mathematics" to be discovered and developed by the ambitious teacher who is thinking of the needs of her pupils. Such a program demands extra time and added preparation for the teacher. This extra work should bring greater results and better equipment for those young Americans who are placing themselves at the feet of the mathematics teacher begging to be instructed, guided and helped in their preparation for the work ahead. Such added efforts will be rewarded by some distant voice saying, "Well done, good and faithful teacher; enter into the joy of knowing that you have done your best."

Flying High

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Why do we not capitalize on the absorbing interest of youth in flying? The armed forces want boys who have a thorough knowledge of the fundamentals of mathematics. Such a foundation can be given through a course in aerial navigation. First, it is assumed that boys who are interested in becoming aviation cadets and girls who wish to be future WASPS are intellectually superior pupils; hence they should have completed two years of mathematics, including elementary algebra and plane geometry, and either have completed or be studying simultaneously intermediate algebra. If possible, a term of plane and spherical trigonometry should be a prerequisite. With such a background pupils are eligible for an elementary course in aerial navigation, which will prove to be a magic carpet transporting both pupils and instructor to the enchanted realm of planes, flights and pilots.

When a former pupil returns with his wings to talk to the class informally about officer training, drift sights, and the clean ship he is flying, members of the class study with keen interest navigation instruments, drag, air resistance and streamlining. When boys bring their model planes to discuss stress and strain, aspect ratio, wing loading, and power loading, they will be unlikely to submit an actual plane to excessive loads due to a steep banking turn, pull out too abruptly from a steep dive, or fly at excessive speeds in rough weather. When solving problems involving center of gravity position, load factors, rate of climb, and gliding angle, the pupil obtains a thorough review of arithmetical computation and trigonometric functions. When discussing lift, drag, and power required, he learns to use logarithms with facility. Controlled experiments have shown that poor results on Army and Navy preinduction arithmetic tests are frequently due to lack of recent use of these processes with resulting forgetfulness. Studying aerodynamics is one method of reviewing arithmetic computation.

After an introductory survey of aerodynamics, the class is ready to consider avigation.[1] The pupil considers first the plane's instruments and learns how to correct instrument readings. Here he uses his knowledge of arithmetic and algebra. For example, he adds two per cent of the indicated airspeed for every thousand feet of increased altitude to the airspeed indicator reading to obtain the true airspeed. In order to follow a true course indicated on the aeronautical chart he corrects for magnetic variation and compass deviation, using positive and negative quantities, to obtain the compass course; then applies a right or left wind correction angle to obtain the compass heading to be followed while flying the plane. A trainer cockpit is used in the classroom, so that pupils can adjust the various instrument settings and then interpret the resulting readings. Woe betide the young man who gives an impossible tachometer reading at takeoff or who neglects to center the ball of the turn and bank indicator when banking the plane for a turn. His companions quickly correct the errors.

Enthusiasm runs high when contact flying begins. Planning the details of a flight from New York to Washington or from Philadelphia to Detroit is a thrilling adventure. Navigating by contact involves a knowledge of map projections and the use of aeronautical charts. Determining the true course and distance from the chart requires measurement of angles and scale use. Distance or time intervals

[1] L. E. Moore, *Elementary Avigation*, D. C. Heath and Company, Boston, 1943.

are used for checking landmarks. The pupil learns how to read the chart intelligently, interpreting correctly the topographic information and aeronautical data. He knows that a racetrack, football stadium, quarry, lake, railroad line, and highway intersection afford valuable landmarks used by the pilot when flying by contact. He is aware of the altitude, visibility, and ceiling minimums which necessitate a transfer to instrument flight. A classroom with one group of pupils planning a contact flight from Harrisburg, Pennsylvania, to Chicago, Illinois, a second group plotting a course from Lakehurst, New Jersey, to Akron, Ohio, and a third group ferrying planes from Lock Haven, Pennsylvania, to St. Louis, Missouri, provides a busy workshop for future pilots.

Dead reckoning is the method of determining the position of a plane by computation. Scale drawing of the triangle of velocities with the wind velocity, true heading and airspeed, true course and groundspeed as the three sides of the triangle will enable the pupil to solve the basic problems of dead reckoning. The first problem is that of finding the groundspeed, compass heading, and time required for a trip before taking off on a proposed flight. The second problem is that of determining the true course, groundspeed, and position of a plane during flight. The third problem is that of checking the speed and direction of the wind, knowing the true course, groundspeed, true heading, and airspeed. The solution of an interception problem provides a review of the proportionality of corresponding sides of similar triangles. The radius of action problem combines algebra and geometry. Algebra is used in the derivation of the formula for radius of action. Geometry is used in solving the triangles of velocities for the flight out and for the flight back. Slide rule computation is involved when substituting the results obtained from the diagram in the formula. Radius of action is the maximum distance a plane can fly out along a course from the point of departure before starting to return on a given amount of fuel. The wind-star problem, that of finding the wind velocity during a flight from two drift sight observations, again offers a review of geometry with the drawing of the airspeed circle and the enclosed triangles of velocity. Practically all dead reckoning problems afford a review of mathematics in some form.

In radio navigation the aircraft is navigated by the use of radio aids, such as, radio ranges, marker beacons, and radio bearings used in obtaining a fix. Pupils enjoy sending and receiving code signals on a telegraph sounder. Future members of the Civil Air Patrol gain practice in radioing such messages as: sighted sub, sank same; released bomb; sighted oil slick; send destroyer, torpedoed seamen. They know the significance of flying the beam, can recognize a cone of silence over a radio range station, and are familiar with the principal orientation methods of determining a plane's position within an *A* or *N* quadrant. Arithmetic is reviewed when obtaining an average bisector used in orientation. Positive and negative quantities are involved in finding radio bearings. Corrections are made for the radio direction finder readings. A radio fix is obtained by plotting true bearings from two radio stations. Radio is extremely useful in the navigation of aircraft.

Celestial navigation is assuming an ever increasing importance as a method of avigation. Checking the geographical position of a plane by taking sextant observations of celestial bodies, using the Almanac to determine the declination and Greenwich hour angle of the observed body for the time of observation, and solving the astronomical triangle for the azimuth and altitude of the star or planet observed will enable the pupil to obtain a line of position for the aircraft by using the computed azimuth and the difference between the corrected observed altitude and the computed altitude. The point of intersection

of two lines of position gives a fix, indicating the position of the plane. The solution of the astronomical triangle gives drill in the use of trigonometric formulas, logarithmic and trigonometric tables. Correcting observations for sextant errors gives a review of geometric relations and arithmetic computation. Plotting the line of position involves geometric measurement. Advancing a line of position parallel to itself along the true course of the plane for a distance equal to the distance covered by the aircraft during the time interval between observations to obtain a running fix reviews arithmetic, algebra and geometry. When three stars are used for observations, three lines of position are obtained forming a "cocked hat." When a pupil determines the fix of an airplane by plotting a "cocked hat," he has applied every phase of high school mathematics functionally, and in addition has obtained a knowledge of avigation in his preinduction work which will augur well for his success in the armed services.

Many educators are fearful of the present trend in emphasis upon technical and scientific subjects. I think that cultural values can be derived from a course in aerial navigation as outlined in a previous issue of THE MATHEMATICS TEACHER.[2] The course in aerial navigation reviews algebraic and geometric concepts, fundamental mathematical principles, and is ideally suitable for attaining the objective stated in the United States Office of Education Committee Report that preinduction training should be functional. Previous testing of enlisted men in the Army has proved that they were unable to apply what they knew. The course suggested would eliminate this difficulty. Pupils would know how to apply scale measurement, interpret graphs, solve formulas, use maps and charts, and solve practical problems involving quantity, measurement, and space relations.

The sequential preinduction program in mathematics recommended by the Army Specialized Training Program might well include:

1st year: elementary algebra
2nd year: plane and solid geometry
3rd year: intermediate algebra; plane and spherical trigonometry
4th year: advanced algebra with elements of analytic geometry and the calculus; aerial navigation.

By placing the course in aerial navigation in the second semester of the fourth year, the pupil gains practice in using his mathematical knowledge functionally just before induction and obtains a practical knowledge of avigation which will be helpful to him as a navigator or pilot. A course in aerial navigation is a practical method of reviewing fundamental mathematical principles and using mathematical knowledge functionally while "flying high."

[2] Lillian Moore, "Aerial Navigation," THE MATHEMATICS TEACHER, XXXV (March, 1942), pp. 99–101.

Learning and/or Mathematics

By Curt Dechert
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How often we hear intelligent people express awe at mathematics. They show great admiration for man's ability to think and express himself in the supposed abstractions of mathematical processes. Therein lies one of the basic reasons for the difficulties that are experienced in learning it. On the other hand, we often hear people recommend mathematics as a beneficial mental calisthenics. This is progress, but I doubt if many realize why this mental calisthenics is beneficial instead of making us mentally muscle bound.

Actually, mathematics *is* the fundamental learning process. Psychologists and educators have not been slow to discover how we learn but mathematicians have not recognized the key to learning while they have held it right in their own hands. Consequently, mathematics has not been measured by the learning process so that the principles of learning could be applied in its teaching.

Man learns in a simple manner. Certain stimuli are recorded and evaluated. These perceptions are related to one another and generalizations or concepts are formed. These concepts permit him to measure new perceptions against the generalizations so that they can be evaluated directly. Also, concepts are related to one another to formulate more basic generalizations. Let's observe this learning process in greater detail so that we can see its mathematical nature.

The most basic stimuli are those we receive through our five senses. We receive the sense of color and form, hardness, odor tone and taste. As more and more of these sensations are experienced we relate them to one another and give them relative or comparative value, i.e., regular or irregular, large or small, high or low, hard or soft, sweet or bitter, pleasant or unpleasant. Thus, in our most basic concepts we have established units of measurement and the number scale—the foundation of mathematics. The number scale, of course, embraces the fundamental processes of arithmetic in that addition, subtraction, multiplication and division are merely counting—relative values of the number scale and a concept of a higher order.

The next step in the learning process is to convert these sensations with comparative value into usable experiences. Here we build equations—a relation of evaluated sensations which are combined to arrive at a conclusion. Then after we have arrived at enough conclusions we can relate them to one another and formulate generalizations—the formula. That is all there is to learning but also that is all there is to mathematics.

Teaching is merely providing the right stimuli for learning and inspiring the evaluation and conceptual progress from these perceptions. How is it then that we forget the learning process when we teach mathematics?

Let us first look at the stimuli that we have to work with. To be easily evaluated and related to other things, the stimulus must be recent, vivid and full of attention value. The most vivid stimuli are those we get directly through our five senses. We can call them *observational.* But we do not have the time or opportunity to receive all of our stimuli directly. Therefore, we must use indirect methods to supply them. The indirect stimuli come to us in several ways. In order of their vividness they are:

1. Pictorial—providing a visual impression of having had the observational experience—
2. Symbolic—a schematic suggestion of the experience—
3. Verbal—a recitation from which the experience could be imagined.

Therefore, when we offer a stimulus, we should offer it in a manner as close to the observational as possible.

Observational stimuli should embrace the use of more than one of the senses wherever possible. Even when the sense of sight is the only contributing factor to the experience, the use of an additional sense is of very definite value. Suppose we look at a book on our desk to discover its title. We can read the title just as easily and accurately without touching it. But, if we pick up the book and read the title the experience becomes more emphatic and we remember it longer. The sense of touch does not contribute to the title of the book, but it adds to the attention value. Its gives us a command of the situation. We can turn the book, lay i: down, or throw it across the room. There is an added measure of importance to the experience that aids the memory and facilitates the forming of an evaluation of the title.

Now we can see the value of building models, constructing figures, making measurements and the various other observational stimuli that can be employed. But we must remember that these things are stimuli for perceptions from which concepts are to be formed. Therefore, we must not fall into the common error of trying to use a stimulus to prove a concept. Let's look at an example: We explain that the 2nd degree equation

$$Ax^2+By^2=C$$

is an ellipse. This is a concept and not a stimulus. Now we use a specific equation and plot it by choosing values for x and solving for y. Thus we have what appears to be an observational stimulus that will prove the concept. However, our stimulus is not directed at the concept of an ellipse because to our untrained eye it is merely an oval. From here on we manipulate the formula for no apparent reason and discover a pair of values we call the foci—another concept. Then we draw an ellipse, plot the foci, and show that every point lies at a distance from them such that the sum of the distances is always constant. This is a stimulus again being used to prove a concept.

How much closer to the learning process we would be if we put the horse before the cart. Let us start by giving the student a piece of string and ask him to fasten the ends leaving plenty of slack. Then he can draw the ellipse and get the attention value provided by his amazement at the regularity of the figure he has formed. Let him retrace the curve to see that every point has a constant sum of its distances from the foci. Now he has a true perception of an ellipse—the only thing that makes it different from an oval. Let him draw more ellipses with different distances between the foci and with different lengths of string. Then he will form a relationship between ellipses. To inspire greater attention and add importance to the perception, show him some practical uses of the figure. Now with these perceptions firmly planted we can begin to add more stimuli to build a concept of the ellipse. With one of our string-made figures before us, we can draw an axis through the foci and one vertical at their midpoint. Thus we get an added perception of the symmetry of the figure. If it is on paper, folding and holding it to the light will add vividness and attention value to the perception. From here we can give a value of "$2a$" to the length of the string and a value of (c, o) and $(-c, o)$ to the foci. Now we choose any point (x, y) on the ellipse and apply the formula for the distance between two points to find its distance from each focus. Adding these distances and setting them equal to the length of the string, we arrive at a true algebraic concept of the ellipse. This simple example is only one of many that could be used to illustrate the value of choosing stimuli as close to the observational end of the scale as possible and using them to progress from perceptions to concepts.

After concepts are clearly formed we can proceed to develop the manipulating skills through exercises without fear of over-

crowding the memory with meaningless routine. But, we must give the student the opportunity to realize his concept. When we look at the value scale of stimuli—

observational–pictorial–symbolic–verbal

we are also looking at the scale of progress from the perceptual to the conceptual.

observational–pictorial–symbolic–verbal
perceptual.................. conceptual

The generalization can best be solidified if the student expresses it in words. Then he will not be illustrating a perception as when drawing a figure or he will not be using a memorized skill as when working a problem. This brings up a teaching device that is almost foreign to the field of mathematics—the theme. "Tell us what you know about the ellipse?" Now we are on the track of grading pupils on what they learn as well as on what they do—on what they understand as well as on what they memorize.

Grading a student on what he understands as well as on what he memorizes provides a teaching opportunity that should be capitalized. Examinations can be planned to appraise the student on both his conceptual progress and skill at manipulating. Furthermore, future assignments can be divided into conceptual drills and manipulating drills. Those students needing conceptual drill can be given a larger share of these assignments while students needing manipulating drills can be given a larger share of those assignments.

There is one more adjunct to the learning process that ties back to the fundamental that "learning is preparation for life." Man is the only animal who depends on learning for his very existence—who will collapse if he relies only on his instincts. This phase of the learning process is his self-appraisal. The student must know what he has learned and have a feel of what he can do with it. Without this, he will not be inspired to carry on even if he has the necessary capabilities. This, again, is on the conceptual side of the learning scale and again requires a verbal or written approach. Thus the theme can become a very important part of the mathematics curriculum.

By tuning the teaching of mathematics to the learning process which it so strongly parallels, it may become much less of a students' stumbling block. And when it does, it will contribute much to the students' capabilities in other fields as well.

Value of Mathematics

Those of us who are charged with the teaching of mathematics in Harvard University take especial note of the fact that the great majority of our first-year students are now studying trigonometry, analytic geometry, and the calculus. In so doing they must master the elements of trigonometry which George Washington used as a young surveyor, and which Thomas Jefferson said were necessary for every man; and in learning the essentials of analytic geometry and the calculus they will be following in the footsteps of Jefferson, who was a devotee of these subjects.

Granted that this widespread election of mathematical courses is to some extent caused by its obvious practical usefulness in a time of national emergency, what is the general basis of the importance of mathematics in our modern world? . . .

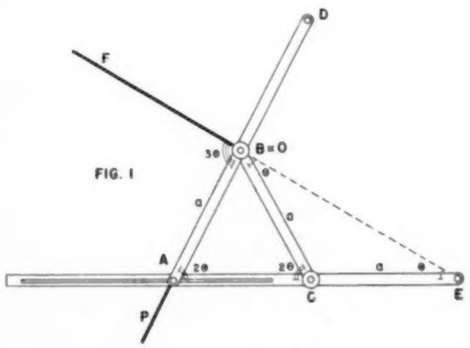
At first the mathematics involved in the arithmetic of the counting house and of land measurement seemed only a humble aid to practical living, but since the time of Archimedes and Plato mathematics has increasingly shown itself the principal weapon of the creative scientific imagination. The world about us turns out to be dominated in its every detail by grandiose patterns of mathematical law, in ignorance or defiance of which individuals and nations may be destroyed. . . .
—Professor George D. Birkhoff, quoted in *Harvard Alumni Bulletin*.

A Grooved Mechanism

By Robert C. Yates
West Point, New York

In an article, *You Can Make Them* (Mathematics Teacher, April, 1942, pp. 182–3), Miss Clara O. Larson lists a few devices as aids in the teaching of geometry. I wish to remark upon some unmentioned properties of several of these devices, not with the hope that my remarks may be transplanted directly into the high school classroom, but that they may serve to deepen the understanding of the teacher.

The first device listed for bisecting an angle (Fig. 1) is an instrument which can be traced back at least to Pascal [1]. Since



its central feature is a straight groove in which the point A slides, it is a much more powerful tool than a simple bisector. The least complicated linkage (compound compasses) that will provide freely the motion of a point along a straight line is composed of five bars connected by pin joints [2, pages 92, 94]. Thus, from this viewpoint, the mechanism of Fig. 1 is equivalent to a linkage of *seven* bars, certainly a complex affair when compared with that natural bisector, the compasses.

If on the grooved bar in Fig. 1 we take $CE=a$ and select the other two bars equal to this length so that $AB=BC=a$, the instrument is a natural *trisector* [3, page 34]. For, if $\angle BEC=\theta$, then $\angle EBC=\theta$, $\angle BCA = \angle BAC=2\theta$ and accordingly,

$$\angle FBA=3\theta.$$

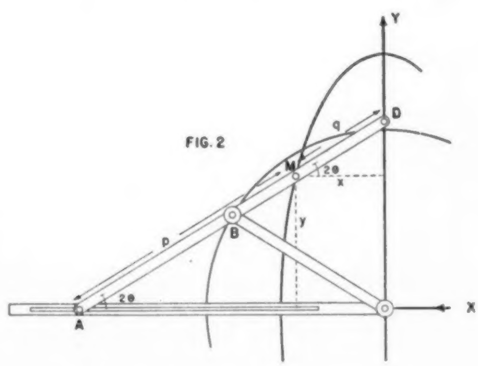
To trisect a given angle FOP, place B upon O and BA upon one side of the angle. Then deform the instrument until E falls on the extension of FB. The third part of $\angle FOP$ is $\angle BEC$.

Other features are worthy of note. If the bar AB be held fixed, the path of E is a *limacon of Pascal.* For, if A be the pole and AB the axis of a polar system with (r, ϕ) as coordinates of E, we have directly,

$$\boxed{r=a(1+2\cdot\cos\phi)}, \quad (\phi=2\theta)$$

a special limacon which is often called the *trisectrix*.

If the bar AB be extended to D such that $BD=a$ and if the grooved bar AE be fixed, *the path of D is a straight line* perpendicular to AE at C. Moreover, *the locus of any point M* (other than A, B, or D) *of AD is an ellipse*. For (Fig. 2), if $M:(x, y)$ is distant p and q units from A and D, re-



spectively ($p+q=2a$ and either p or q may be negative which has the effect of extending the bar AD in either direction), then

$$\boxed{\begin{gathered} x=p\cdot\cos 2\theta, \quad y=q\cdot\sin\ 2\theta \quad \text{or} \\ x^2/p^2+y^2/q^2=1 \end{gathered}}.$$

This family, for various points M, includes

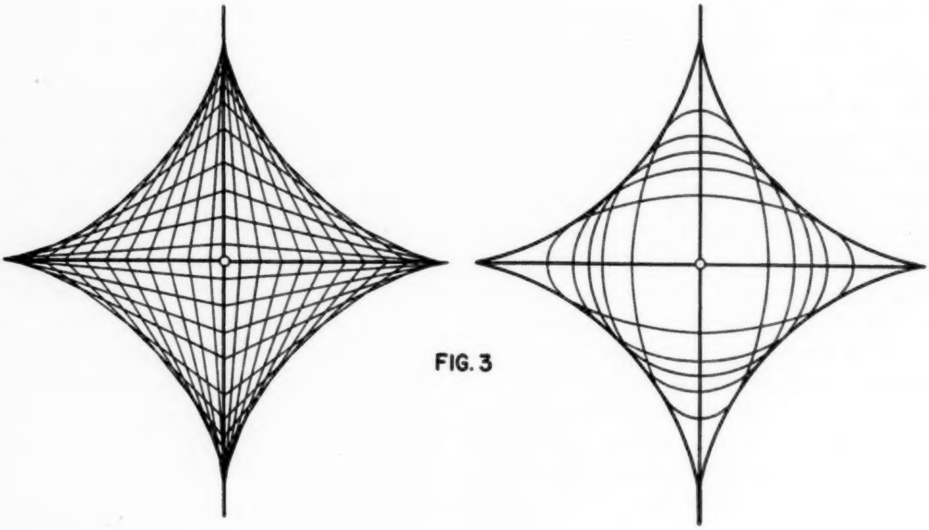


FIG. 3

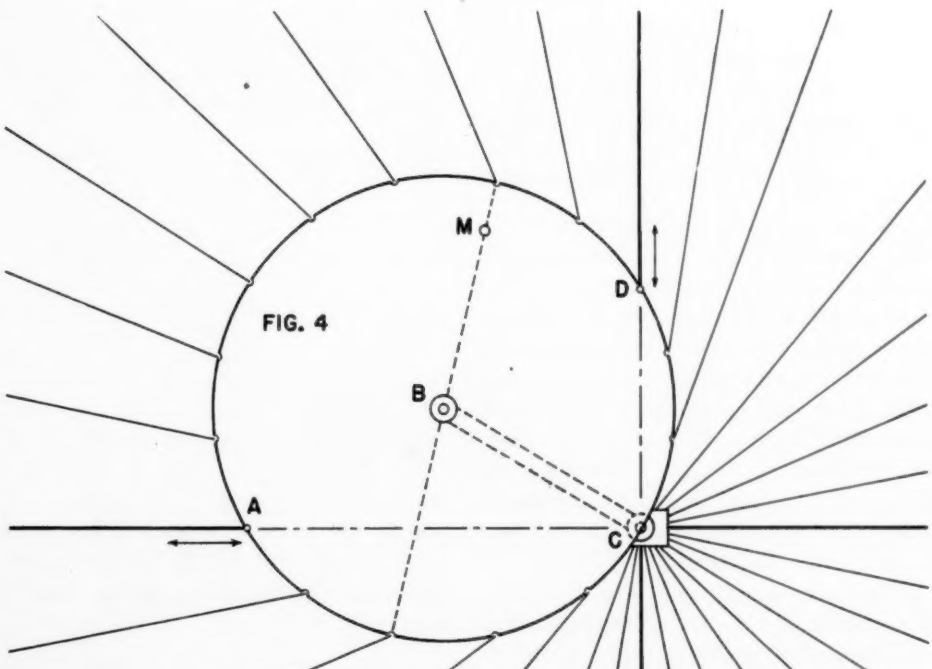


FIG. 4

the two reference axes as degenerate members. The midpoint B of AD, of course, describes a circle. This will be recognized as the familiar Trammel of Archimedes which has an intimate connection with the trisection problem [3, page 150]. Any point of a ladder leaning against a vertical wall describes an ellipse if the ladder falls in a vertical plane.* Well known is the fact that as the bar AD (the ladder) moves with its ends in contact with two fixed per pendicular lines it envelopes an astroid, the hypocycloid of four cusps. For (Fig. 3), the line along which the bar (of length $2a$) lies at any time is

$$x/\cos 2\theta + y/\sin 2\theta = 2a.$$

The envelope of this family, of parameter 2θ, is produced with the derivative:

$$-x\cdot \sin 2\theta/\cos^2 2\theta + y\cdot\cos 2\theta/\sin^2 2\theta = 0,$$

to give

$$\boxed{x = 2a\cdot\cos^3 2\theta, \quad y = 2a\cdot\sin^3 2\theta}.$$

This astroid is also, of course, the envelope of the family of ellipses just mentioned. Its equation is obtained by eliminating p and q between the equation of the ellipses and the derivative:

$$x^{4/3}/p^2 - y^{4/3}/q^2 = 0.$$

We have thus:

$$\boxed{x^{2/3} + y^{2/3} = (2a)^{2/3}}.$$

The features of this mechanism are considerably enhanced if the bar AD be replaced by a circular disk (Fig. 4) of radius a with its center attached to B and any point A of its rim constrained to the groove.† The diametrically opposite point D, as mentioned previously, moves along the line DC perpendicular to AE. Since A and D are the extremities of an arbitrarily selected diameter, the action is the same for any diameter. Thus *every* point on the rim of the disk describes a straight line through C and these form pairs of perpendicular lines generated by extremities of diameters. It is evident, furthermore, from the discussion of Fig. 2, that any point M of the disk (or any point M rigidly attached to the disk) traces an ellipse which is symmetrical with respect to the two lines described by the extremities of the diameter through M [2, page 90].

It seems at first blush somewhat strange that the mechanism under discussion should have any connection with roulettes. But consider a circle of radius a rolling inside a fixed circle of radius $2a$ (Fig. 5). A

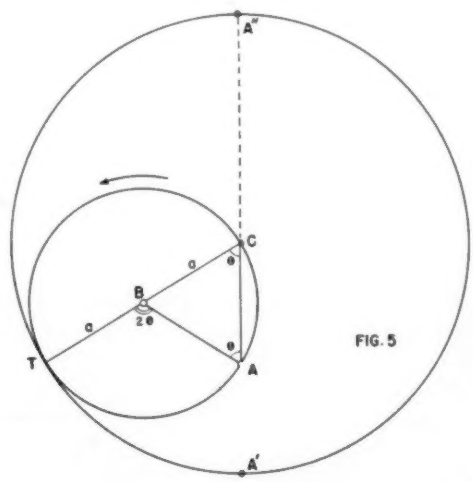


FIG. 5

point A on the rolling circle starts at A'. Since arc AT = arc AT', evidently $\angle TBA = 2(\angle TCA')$. But $\angle TBA = 2(\angle TCA)$. Accordingly, C, A, and A' are collinear and the path of A is thus the diameter $A'A''$. The action here then is precisely that of Fig. 4 [2, page 90]. Any point of the disk, or any point rigidly attached to the disk, describes an ellipse. Some toy tops with colored disks found in dime stores,‡ physics and psychology laboratories, are nice illustrations.

* This is so only if top and bottom remain in contact with wall and floor. Actually, the top of the ladder in a free fall leaves the wall one-third of the way down [4, page 318].

† This in no way complicates Miss Larson's devices. The bars might just as well be plane inextensible sections (or plates) upon which straight lines are scored for identification purposes.

‡ At least before priorities.

Further comments on the mechanism of Fig. 1 seem necessary. Let the point E be fixed and B be moved along a fixed line BE. Then the point A describes the *cycloidum anomalarum* of Ceva [3, page 29], a curve resembling the lemniscate ("horizontal figure-of-eight"). This curve was proposed by Ceva as an aid in solving the trisection problem.

If the arrangement of Fig. 1 be reflected in the line AC, the result is the ordinary *pantograph* built from a rhombus with two extended legs and a grooved diagonal (Fig. 6), which is the second of

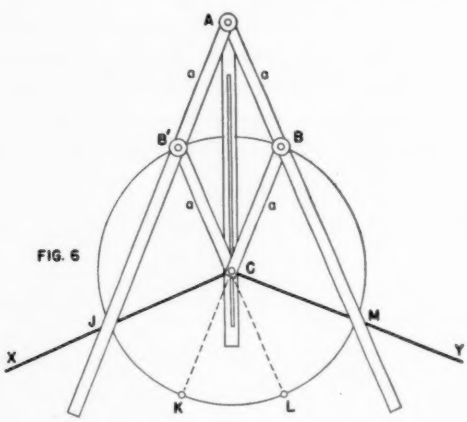


FIG. 6

Miss Larson's bisectors. It too has trisecting possibilities [3, page 35], [5], [6]; upon a given angle XCY place the joint C of the pantograph at the vertex with the grooved bar along the bisector of the given angle. The circle, center C and radius a, meets the sides of the angle in J and M. Let A move along the bisector* until the extended sides, AB' and AB, pass through J and M, respectively. Then

$$\boxed{\angle B'AB = \angle B'CB = (\tfrac{1}{3})(\angle XCY)}.$$

For, arcs JK, LM, and $B'B$ are equal since they are included between parallel chords of the circle and thus $\angle JCK = \angle LCM = \angle B'CB = \angle KCL$.

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* If A is moved along the bisector the grooved bar plays no role, in fact, it is cumbersome extra baggage. A much better arrangement would result, if we must have a groove, by fixing C to the bar and allowing A to move in the slot.

Snow White and the Seven Dwarves*

(A Mathematical Skit)

By Alice E. Smith
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Characters: Snow White, a new Senior and very bright.
Queen, an old Senior, who had previously been the top ranking student.
Prince Charming, a college man of great learning.
The Dwarves: "Doc" Formula
Grumpy Fraction
Sneezy Percent
Happy Ratio
Sleepy } the Congruent triangles
Bashful }
Dopey Decimal
Mirror

Act I

(*Queen enters and crosses stage to stand in front of mirror.*)

Queen: Mirror, mirror on the wall,
Who is the smartest one of all?
Shining glass in which I gaze,
Has anyone else here got four A's?

Mirror: Noble Senior, you're the tops
Of past and present student crops.
You're the smartest one there is.
To be quite frank—you are a whiz.

Queen: I know I beat the school last year;
The old girls give me naught to fear.
But tell me, mirror, tell me true,
What bright *new* girls have come to view?

Mirror: Flash! I've got the latest news,
A girl will step into your shoes.
You *were* the smartest one in sight,
But now *beware*, beware Snow White!

(*Queen stamps angrily across the floor. Snow White enters and walks timidly across the stage.*)

Snow White: They said you could help me.

Queen: Let's see your program. Ah-hah! Snow White! And what is your trouble, my *dear?*

Snow White: I'm trying to plan my program. I know I have to take English, and Biology, and French, but what else shall I take?

Queen: What kind of a student are you?

Snow White: Well in my last school I always got A's, but I hear that this school is much harder.

* Presented in a Mathematical Assembly at Dana Hall School on November 6, 1942.

QUEEN: Then what you need is a snap course, so something will be easy. I suggest Mathematics III—that's a cinch! (aside) I ought to know. That's how I got four A's, by *not* taking it!

SNOW WHITE: Oh, thank you. I'll go tell the schedule committee right away. (*exit*)

QUEEN: (*to the mirror*) That will fix the little minx,
She'll find it harder than she thinks.
Won't she be glad of my insistence,
When she's involved in "Rate-Time-Distance"?

(Curtain)

ACT II

(*Snow White is studying her homework.*)

SNOW WHITE: I don't see how *anyone* could possibly say this was a *snap* course. Maybe, it's because she's so awfully bright, and everything's a snap. How will I ever learn all these awful things! If it were just Algebra or Geometry, I wouldn't mind, but it's filled with *Arithmetic!* Ugh! How I despise decimals and fractions, and I never did understand per cent.

(*Enter dwarves, all but Dopey.*)

DWARVES: Did you mean us when you said all those things?

SNOW WHITE: Well, who are you?

DOC FORMULA: Let me introduce us. I'm Doc Formula. (*Bows deeply.*) And this is Grumpy Fraction.

GRUMPY: (*sourly*) Hullo.

SNOW WHITE: Why is he so Grumpy?

DOC: Someone has been spreading rumors about his family. Called them *improper* fractions.

SNOW WHITE: Oh, how dreadful.

DOC: Next we have Happy Ratio. (*He bows.*)

SNOW WHITE: Why are you so happy?

HAPPY: Wouldn't you be when you're in the news all the time? Why they're rationing everything, now. Of course once in a while I'm down in the dumps, but that's only when I'm all out of proportion.

DOC: And this is Sneezy Percent.

SNOW WHITE: Sneezy? Why Sneezy?

SNEEZY: Doesn't it sound like it. Percent. Percent. (*Sneezes each word.*)

DOC: And these are the twins, Sleepy and Bashful, the Congruent Triangles.

SLEEPY: (*yawns*) I can sleep whenever I want to, now. He's just like me, so why should both of us stay awake? (*Yawns again.*)
Bashful acts it and says nothing.

SNOW WHITE: But shouldn't there be seven of you? I can count only six.

GRUMPY: Dopey must be lost again. (*He goes outside and drags him in by the ear.*) Come here and meet Snow White. This is Dopey Decimal.

DOPEY: Hullo, hullo, hullo.

SNOW WHITE: Why do you call him Dopey?

DOC: 'Cause he's always repeating himself.

DOPEY: I'm worried. I'm worried.

SNOW WHITE: Why shouldn't you worry? We all worry about you.

DOPEY: I heard some sophomores talking about rounding off decimals, and I'm scared. I'm scared. Next thing you know I'll be thrown away completely.

DOC. You seemed to be having trouble, Snow White, when we came in. I'm a doctor, and I've got formulas for everything. What's wrong?

SNOW WHITE: It's just this Math. I *can't* do Arithmetic. And if I can't do Arithmetic, I won't pass my Math. And if I don't pass Math, Prince Charming won't love me.

HAPPY: He won't?

SNOW WHITE: No, he's a Math genius. He teaches aviators to work airplane problems and helps engineers build bridges, and scientists invent new things, and tells the Navy how to run their ships. He knows *everything* about Math, and if I flunk it, he'll never look at me again.

DOC. Well, don't worry. We'll help you. If we're your friends, nothing in Math can bother you.

DWARVES: (*sing*) Heigh ho, heigh ho,
It's off to work we go,
To make things bright
For sweet Snow White,
With a heigh, heigh ho;
Heigh ho, heigh ho, heigh ho,
We'll say, "I told you so."
When you have made
The highest grade,
With a heigh, heigh ho!

(Curtain)

ACT III

(*Snow White is taking her final examination in Math III.*)

SNOW WHITE: The first part of that exam wasn't so bad. Now if I can only manage to do these problems, everything will be all right.

(*Enter the Queen, disguised as a Rate-Time-Distance problem.*)

QUEEN: I am your first problem. (Reads) "What is the rate of speed in feet per second at which a bullet travels if a marksman hears his bullet strike a target 2200 feet away, six seconds after the discharge? Sound travels 1100 feet per second." (*Hands problem to Snow White and exits laughing.*)

SNOW WHITE: Oh, heavens, that sounds terrible. How can I ever do that?

(*Enter Grumpy and Doc.*)

DOC: Now don't let that problem bother you. Just use one of my formulas.

SNOW WHITE: But that gives me fractions.

GRUMPY: Just remember those fractions are *my* family. You're not afraid of us, are you?

(*Both watch over her shoulder while she works.*)

SNOW WHITE: That wasn't so bad. I wonder what's next?

(*Enter Queen, disguised as a mixture problem.*)

QUEEN: A milkman has 1000 quarts of milk that test 4.7% of butter fat, but the city in which he sells his milk requires only 3% butter fat. How many quarts of cream testing 23.2% butter fat may he separate from the milk and still meet the city requirements?

SNOW WHITE: Percentage *and* decimals! Oh, if only Sneezy and Dopey were here to help me out.

(*Enter Sneezy, pulling Dopey.*)

SNEEZY: Kind of chilly here. Percent! Percent!
DOPEY: God bless you. God bless you.
SNEEZY: Did you say you wanted us?
SNOW WHITE: Yes, I'm stuck with percent.
SNEEZY: Now you know all my family. We're not a bit stuck up.
SNOW WHITE: Then it must be the decimals that bother me.
DOPEY: But you know I'm harmless, harmless. Just forget I'm even here. Move me over if you want.

(*They stand by her and watch her work it out.*)

SNOW WHITE: Well, that's done. Just one more.

(*Enter Queen, disguised as a Geometry problem.*)

QUEEN: If D and E are midpoints of sides AC and BC of triangle ABC, and if DE is prolonged its own length to F, prove that AC:FB::BC:CE.
SNOW WHITE: Oh, where will I ever start on this one? It seems all mixed up.
BASHFUL and SLEEPY: (*entering*) What about trying us? We're easy to do and ought to help you out.
HAPPY: And don't forget me. If I had my twin along, we could be that proportion. I'll stand by this mirror and you'll think you have two equal ratios.
SNOW WHITE: Why, this problem practically does itself. (*Writes busily.*) That finished my exam. (*She takes it off to one side to pass it in, and then returns.*)

(*Enter Prince.*)

PRINCE: Ah, Snow White, they said you had just finished your exam and could see me. How was it?
SNOW WHITE: Well, I hope I did well. You see, it was Mathematics.
PRINCE: If you pass, Snow White, I want you to wear my fraternity pin.
SNOW WHITE: Which one? You belong to so many honorary fraternities. You're *so* brilliant.
PRINCE: Take your pick, or take them all.

(*Enter Queen.*)

QUEEN: Mirror, mirror, on the wall,
Who *now* is the smartest one of all?
Tell me, have I reached my goal?
Shall *I* lead the honor roll?
MIRROR: Smart you are, but not enough.
She is made of better stuff.
You get A's without a fuss,
But Snow White's marks are all A+!

(*Snow White selects a fraternity pin from Prince and pins it on her own dress.*)

Curtain

Horner's Method and the Algorithm for the Extraction of Square Roots and Cube Roots

By Nelson B. Conkwright
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It seems not to be generally known that the algorithms for the computation of the square root and the cube root of a number N differ from Horner's procedure for the solution of the equations $x^2-N=0$ and $x^3-N=0$ only in the formal arrangement of the work. It is the purpose of this discussion to show how the resemblance between the two methods of computation can be utilized in the extraction of roots by means of the algorithm.

The routine procedure which will be developed is similar to certain devices explained in some of the older algebra texts. But it is believed that the discussion offered here will constitute a more effective and stimulating explanation of these devices.

The nature of the ideas to be developed can perhaps best be explained by the presentation of a concrete numerical illustration, rather than by an abstract argument. Let it be required to find the square root of 7. The standard procedure (with some ciphers and decimal points inserted to facilitate comparison with Horner's process) is as follows:

		2. 6 4 5 7
	7. 4.	
4. 0 6	3. 0 0 2. 7 6	
5. 2 0 4	0. 2 4 0 0 0. 2 0 9 6	
5. 2 8 0 5	0. 0 3 0 4 0 0 0. 0 2 6 4 2 5	
5. 2 9 0 0 7	0. 0 0 3 9 7 5 0 0 0. 0 0 3 7 0 3 4 9	
5. 2 9 1 4 0	0. 0 0 0 2 7 1 5 1	

It is thus seen that $\sqrt{7}=2.6457+h$, where h is between 0.000 05 and 0.000 06. We inquire then as to how accurately the root can be obtained by taking

$$(1)\qquad \sqrt{7}=2.6457+\frac{R_5}{D_5}$$
$$=2.645\ 751\ 311\ 562\cdots,$$

where R_5 and D_5 are the last remainder and trial divisor respectively in the tabulated computation above. The fact is that the true value of the root differs from the number in the last member of (1) by not more than

$$\tfrac{1}{5}(0.000\ 06)^2=0.000\ 000\ 000\ 72.$$

This follows from the fact that if we compute $\sqrt{7}$ by using Horner's method to solve the equation $x^2-7=0$, it will be found that the successive reduced equations are

$$x^2+D_jx-R_j=0, \qquad j=1,2,3,\cdots,$$

where D_j and R_j are the jth trial divisor and the remainder respectively which appeared in the use of the algorithm. Thus h is the root between 0.000 05 and 0.000 06 of the equation

$$(2)\qquad x^2+5.2914x-0.000\ 271\ 51=0,$$

and consequently

$$\left|h-\frac{R_5}{D_5}\right|=\frac{h^2}{5.2914}<\tfrac{1}{5}(0.000\ 06)^2$$
$$=0.000\ 000\ 000\ 72.$$

It follows that equation (1) gives a value of $\sqrt{7}$ in which the digits in the first eight decimal places are certainly correct. There is doubt as to whether the number in the ninth decimal place should be 0 or 1.

It can be shown by means of equation (2) that the true value of h is *less* than R_5/D_5.

The facts which have been pointed out are not peculiar to the number 7. The same procedure can profitably be applied whenever a square root has already been computed to several decimal places. It is quite easy to show that in any case the algorithm differs from Horner's method only in the formal arrangement of the work.

The routine procedure for the general case may be formulated as follows: Assume that the first r decimal places in the square root of a number N have been found. Let R and D denote the remainder* and trial divisor respectively which would be used to find the next digit in the root. If the digit in the $(r+1)$th decimal place is found to be s, then we may take that part of the root subsequent to the rth decimal place to be R/D with an error not greater than

$$\frac{[10^{-(r+1)}(s+1)]^2}{D}.$$

Equation (2) can be used to find a still better value of $\sqrt{7}$. Let

$$h=a+k, \tag{3}$$

where $a=0.000\ 051\ 31$. Then $k<0.000\ 000\ 002$. Upon substituting $a+k$ for x in (2), and dropping certain terms which would not affect the result in the first twelve decimal places, it is found that

$$h=a+k=\frac{0.000\ 271\ 51-(0.000\ 051\ 31)^2}{5.2914}$$

$$=0.\ 000\ 051\ 311\ 064$$

and $\sqrt{7}=2.645\ 751\ 311\ 064$

with the error beyond the twelfth decimal place.

This device can of course be employed in computing the square roots of other numbers.

The methods which have been explained can be extended in an obvious manner to facilitate the determination of cube roots. For example, in computing the cube root of 15, it is found that this root is $2.466+h$, and that the trial divisor and remainder which would be used to find the next digit are 18.243 468 and 0.003 869 304 respectively. The number h is the root between 0.000 2 and 0.000 3 of the equation

$$\begin{aligned} x^3+7.398x^2+18.243\ 468x \\ -0.003\ 869\ 304=0. \end{aligned} \tag{4}$$

(The coefficient of h in this equation is three times that part of the cube root already found.) If we suppress the first two terms in (4), and determine h by solving the resulting linear equation, the error in the value of h thus found will not be greater than

$$\left.\frac{x^3+8x^2}{18}\right]_{x=0.003}<0.\ 000\ 000\ 05.$$

Thus we obtain a value of $\sqrt[3]{15}$ with the error in the eighth decimal place.

* To obtain this remainder, all periods of N which do not consist entirely of zeros must be "brought down."

Arithmetic Attitudes*

By C. W. Hunnicutt
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Arithmetic has too often been taught as a skill unrelated to life outside the classroom. Children are shown few everyday places to apply their skill. A common complaint is that problems are unrealistic and deal with experiences remote from the lives of children. There is likely to be insufficient attention to the use people constantly make of arithmetic concepts and relations. If arithmetic is to be fully meaningful and not quickly forgotten, greater care must be taken to assure understandings that function in daily life.

In New York City a controlled experiment throughout a six-year period enabled a large group of children to experience an "activity" program, while comparable children attended schools committed to a "regular" pattern of teaching. In a Survey[1] of the outcomes, a test entitled *What I Believe and Do*[2] was among the ones given to 1135 upper sixth grade children of four matched pairs of schools committed respectively to the two programs. Two items dealing with the above problem were included in this test. The children were asked to respond "Yes," "No," or "Unable to answer" to the questions: "During the past month have you used any arithmetic outside of school? (Don't count homework)" and "Do you use the things you learn in arithmetic for any purpose other than school work?"

It was possible to analyze results within several categories, dividing both by sex and by the type of program to which the schools were committed. Moreover, the scores of children in the academically best and the academically poorest class of this grade within each school were segregated for further analysis. All the comparisons here reported depend upon the use of the Chi Square technique to determine the levels of confidence of statistical significance. All percentages are reported to the nearest whole per cent.

Approximately two-thirds of the children of all groups were aware of using arithmetic outside the classroom. However, more children in the "Activity" schools (69%) than in the "Regular" schools (64%) replied "Yes." This difference yielded a level of confidence below 0.01, indicating high significance.

The difference between boys of the "Activity" (67%) and of the "Regular" (61%) schools was slightly greater than between the girls of the two programs (72% and 67%, respectively). This also reveals that the girls of the respective programs were apparently more likely than the boys to be aware of the use outside school.

As might be expected, the children in the academically better classes were far ahead of those with lower ratings in reporting such usage. The percentages were 72% and 56%, respectively.

The above data indicated that there were rather large numbers of sixth grade children who reported using arithmetic only in school work, that this was more true with boys than with girls, that children in brighter classes were more aware of usage than the slower children, and that the activity pattern of teaching has increased children's awareness of arithmetic usages outside of the school.

* Responses of Children in "Activity" and "Regular" Schools in New York City.

[1] New York State Education Department. [Directed by Morrison, J. Cayce] *The Activity Program. A Curriculum Experiment.* Brooklyn, New York: Superintendent of Schools, 110 Livingstone Street, 1941. 182 pp. Paper Covers. $1.00.

[2] A description of this test and its results may be found in the above report. Samples may be obtained from the author, Syracuse University, Syracuse, New York.

◆ THE ART OF TEACHING ◆

A Meaningful Unit in Indirect Measurement

By Vincent J. Glennon
State Teachers College, Fitchburg, Massachusetts

In his *Teaching Arithmetic in the Elementary School** Robert L. Morton says with reference to the topic of Mensuration and Intuitive Geometry:

> The arithmetic of mensuration has too often been taught by a method which is based upon the drill theory. It provides excellent opportunities for the use of methods which are based on the meaning theory.

Within recent years much has been written with the same thought in mind—that of emphasizing the need for teaching arithmetic through meaningful experiences. In this article I am outlining briefly a unit of work that can be carried out with pupils on the eighth grade level.

Aims

A. General

The main aim in carrying out this unit is to develop on the part of the pupil a functional knowledge of the tangent ratio as a means of indirect measurement.

B. Specific

1. To show the importance of the triangle in life.
2. To show the relationship between any two sides of a right triangle.
3. To formulate a table of tangent ratios. (Two decimal places.)
4. To compare the table with a standard table of tangent ratios.
5. To be able to use in a meaningful way such terms as tangent, right triangle, degree, hypotenuse, legs, adjacent, and transit.
6. To construct an instrument for use in sighting and measuring angles of elevation.
7. To develop the ability to sight and measure angles of elevation.
8. To develop the ability to compute the height of a building, tree, flagpole, using angle measurement and a table of tangent ratios.
9. To watch the city engineer while he computes the height of an object using the transit.
10. To compare our computation with that of the engineer.
11. To discuss the reasons for any differences between the computation of the engineer and our own computation.
12. To make a scale drawing of the measuring carried out using the drawing board, T-square, triangle and protractor.

Approach

The approach to this unit evolves directly out of a definite interest on the part of the pupil. Many times pupils ask the instructor the height of the school building or the height of the flagpole in front of the school. Pupils have been found to discuss the same question among themselves. When asked by the instructor how they would proceed in finding the height of the object, pupils suggest such methods as "use a ladder," "drop a rope from the top, then measure the rope."

When the instructor suggests that there is a simpler and more accurate method, the pupils are eager to hear about it and discuss it. The learning activities of the unit then are initiated in a very natural meaningful situation.

* Morton, Robert Lee, *Teaching Arithmetic in the Elementary School,* Silver Burdett Company, New York, Vol. III, Upper Grades, 1939.

Learning Activities

1. Through pictures, lantern slides, newspaper and magazine clippings show the use of the triangle in architecture and structural work.

2. Plan bulletin board and oral reports that will show the use of the triangle in life.

3. With a hinged angle and a third side (yardstick) show that as the angle opens the length of the side opposite the angle increases. Also, when angle is 45°, show that the side opposite is equal in length to the side adjacent.

4. Using the protractor, pupils construct a right triangle with angle of 10°. With ruler pupils measure the length of the side opposite the angle and the length of the side adjacent to the angle. They divide the length of the side opposite by the length of the side adjacent and establish (to two decimal places) an approximate tangent ratio for an angle of 10°. The same procedure is used to establish tangent ratios for angles of 20, 30, 40, 50, 60, 70, and 80 degrees. The pupils then have formulated a table of their own.

5. Pupils compare their table of ratios with a standard table passed out to them in mimeographed or hectographed form.

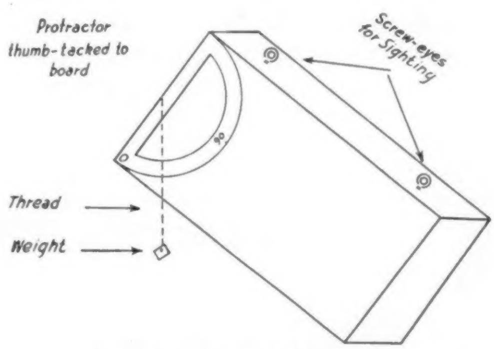
6. Pupils discuss the reasons for the differences existing between their tables and the standard tables.

7. In industrial art classes boys construct an instrument for use in sighting the height of an object and measuring the angle of elevation. This instrument can be made with block of wood, protractor, length of thread and a small weight (see drawing).

8. Before going into the school yard for the field work, the class is divided into groups of three pupils each. Pupil *One* is assigned the job of measuring with a tape the distance from the observer to the base of the object. Pupil *Two* is responsible for sighting the object and measuring the angle of elevation. Pupil *Three* records the data in notebook.

In order that each pupil will have experience in each assignment, the three pupils within a group exchange assignments making three calculations at varied distances from the same object.

9. After all necessary measurements in the field have been made, the groups re-



Instrument for Measuring
Angle of Elevation
Stock $\frac{3}{4}'' \times 3'' \times 9''$

turn to the classroom to do the computation work. The pupils within a group work individually. When all three members of a group have finished the computation work individually, they compare the results of their work. Errors are noted and corrections made.

10. The city engineer is invited to the school to discuss civil engineering as a profession; to explain the function of the transit; and to demonstrate the use of the transit in finding the height of the school building or some other object.

11. Pupils compare their computation with that of the civil engineer.

12. Pupils discuss possible reasons for any differences between their answers and that of the engineer.

13. As a culmination to the unit each pupil makes a scale drawing of the measuring he carried out. This involves the use of the protractor, T-square, triangle and drawing board.

These papers can then be used to form a very interesting display of pupils' work.

Outcomes

A. Subject matter learnings

1. A functional knowledge of the use of the triangle in indirect measurement.
2. Increased vocabulary power (*tangent, opposite, adjacent, ratio, transit, right angle, right triangle, degree, hypotenuse, legs, elevation, scale drawing*).
3. Ability to measure angles and compute distances by use of the tangent ratio.
4. A knowledge of scale drawing.

B. Concomitant learnings

1. Strengthening of desirable school citizenship habits—cooperation, self-control, respect for work of others.
2. Increased power in desirable work habits—neatness, accuracy, checking work, and suspended judgment.

(When computations are completed, pupil must add his eye-level height to the answer to get the correct answers.)

EDITORIALS

Importance of Education in a Democracy

A PART of a recent letter from Professor Harl R. Douglass, Director of the College of Education, at the University of Colorado, is worthy of the attention of all of our readers. He said in part:

> This is the time, of course, when schools should be laying the foundations for intelligent public attitudes upon the great problems with which we will be faced in the next twenty-five years, namely, (1) those having to do with national economic reorganization; and (2) those having to do with the international relationships and peace. Those who waged the battle for free public schools had in mind such a purpose, as is indicated by hundreds of statements that have been made by our presidents and national leaders from George Washington on down.

He then included two quotations that will be of interest here:

> Promote then, as an object of primary importance, institutions for the general diffusion of knowledge. In proportion as the structure of a government gives force to public opinion, it is essential that public opinion should be enlightened." (George Washington in his Farewell Address.)
>
> Education is more indispensable, and must be more general, under a free government than any other. In a monarchy, the few who are likely to govern must have some education, but the common people must be kept in ignorance; in an aristocracy, the nobles should be educated, but here it is even more necessary that the common people should be ignorant; but in a free government knowledge must be general, and ought to be universal. (John Q. Adams.)

These quotations are illustrative of many that might be given to indicate how in a democracy more than in any other form of government the people need to be educated. This can be done better in the future by paying greater attention to two things:

1. The need for better education of the gifted pupils in the schools who are now the most retarded and who subsequently should be the leaders in the days ahead.
2. The need for developing a higher type of intelligent followership among those pupils who obviously will not be leaders.

A most stimulating article has just appeared by Professor Seashore.* It should be read by every classroom teacher because as he says:

> *It is the function of the educator to keep each individual busy in wholesome training at his highest natural level of successful achievement in order that he may be happy, successful, and good.*
>
> The greatest contribution of psychology to education is the discovery of the nature and magnitude, the relative fixity and modifiability, and the imposing significance of the established facts about individual differences. . . . The principle of individual differences is a central theme of modern psychology in the classroom. So it is also in courses on education. But, strange to say, actual observance of such facts in teaching has remained largely like the gusts of wind about which we talk but do nothing.

As Professor Douglass further said:

> The teacher or administrator today who is not reading widely and intensively in these areas is hardly capable of preparing young people for the world in which they will live. They are schoolmarms and schoolmasters rather than modern educators. Failing to keep informed and oriented in these areas is inexcusable in the light of the large number of short, readable, reliable books and pamphlets that are available.

We submit below a list of selected small books and pamphlets which Professor Douglass and we think will illustrate what he has in mind.

W. D. R.

I. SHORT NON-TECHNICAL RELIABLE BOOKS ON POST-WAR PROBLEMS

Agar, Herbert, *A Time for Greatness*, New York, Simon and Schuster, 1943.

Chase, Stuart (1941–42), Goals for America; The Road We are Traveling; The Dollar Dilemma; Tomorrow's Trade; Farmer, Worker, Businessman; and Winning the Peace. Six small dollar monographs. Twentieth Century Fund, New York City.

* Seashore, Carl E., "An Educational Decalog," *School and Society*, November 6, 1943.

Davies, Joseph, *Mission to Moscow*, New York, Simon and Schuster, 1941.

Hoover, Herbert and Gibson, Hugh, *The Problems of a Lasting Peace*, Garden City, N. Y., Doubleday, Doran, 1942.

Hindus, Maurice, *Mother Russia*, New York, Doubleday Doran, 1942.

Lippman, Walter, *American Foreign Policy*, Boston, Little, Brown & Company, 1942.

Marshall, James, *The Freedom to Be Free*, New York, John Day Company, 1943.

Motherwell, Hiram, *The Peace We Fight For*, Harper and Brothers, 1943.

Motherwell, Hiram, *Rebuilding Europe After Victory*, Public Affairs Pamphlet No. 81, Public Affairs Committee, New York, 1943. 10¢. Pp. 32.

Rugg, Harold O., *Now is the Moment*, Houghton, Mifflin, 1943.

Van Doren, Mark. *Liberal Education*, Henry Holt and Company, New York, 1943. Pp. 186. Price, $2.50.

Wallace, Henry, The Price of Free World Victory, New York, Fisher, 1942.

Wallace, Henry, *The Century of the Common Man*, New York, Reynal and Hitchcock.

Willkie, Wendell, *One World*, New York, Simon and Schuster, 1942.

Willkie, Hoover, Gibson, Wallace, Welles, *Prefaces to Peace*, New York. Book of the Month Club. Symposium: One World, Problems of Lasting Peace, Price of Free World Victory, Blueprints for Peace.

Yutang, Liu, *Between Tears and Laughter*, John Day Co., New York.

II. Pamphlets and Periodical Articles
Post-War Economic and Social Problems

Hansen, Alvin H., "After the War—Full Employment," *National Resources Planning Board*, January, 1942.

Bailey, Thomas A., *America's Foreign Policies, Past and Present*. Headline Books, No. 40. The Foreign Policy Association: New York, 1943, 25¢. Pp. 96.

Dean, Vera Micheles, *The Struggle for World Order*, Headline Books, No. 32, The Foreign Policy Association, New York, 1941, 25¢. Pp. 96.

Educational Policies Commission, *Education and the People's Peace*. National Education Association of the United States: Washington, D. C., 1943, 10¢. Pp. 59.

National Resources Planning Board, *After the War—Toward Security*. Washington, D. C., Superintendent of Documents, 1942.

The United States in a New World. A Study and Discussion Outline and reprints of very splendid reports: I, Relations with Britain; II, Pacific Relations; III, The Domestic Economy; IV, Relations with Europe. By the Editors of Time, Life and Fortune. Bureau of Special Services, 9 Rockefeller Plaza, New York City.

Important References on War Material for Teachers of Mathematics and Science

So long as the war continues teachers of mathematics and science will still wish to have suggestions as to what material may be helpful to them. The list of such material is too long to be included here, but the following list should be of some practical value and it is up-to-date and easily accessible. W. D. R.

Highschool Science and Mathematics in Relation to the Manpower Problem. 1943. 32 p. Free. Robert Havighurst, University of Chicago.

Mathematics in the Victory Program—Victory Corps Series Bulletin 6. 1943. 7 p. Free. Superintendent of Public Instruction, Des Moines, Iowa.

The Teacher of Mathematics and the War Savings Program—problems of elementary and highschool classes in math and business arithmetic. 1943. 43 p. Mimeographed. Free. Education Section, War Finance Div., Treasury Dept., Washington 25, D. C. See also The Mathematics Teacher for December 1943 which contains this material. Reprints from The Mathematics Teacher may be obtained for 20¢ each postpaid by writing to The Mathematics Teacher, 525 West 120th St., New York 27, N. Y.

Teaching Aids—sample problems and examinations from courses taught at Naval Training Schools and Aviation Bases, covering aeronautics, communications, navigation. 1942. 63 p. Free. Bureau of Naval Personnel, Navy Dept., Washington, D. C.

Technical Manual—Army Arithmetic. War Dept. TM 21-510. Simple, concrete problems. May 1943. 48 p. 20¢. (Also *Technical Manual—Army Reader*. TM 21-500. May 1943. 145 p. 35¢.) Supt. Doc.

◆ IN OTHER PERIODICALS ◆

By NATHAN LAZAR
Midwood High School, Brooklyn, New York

The American Mathematical Monthly

October, 1943, vol. 50, no. 8.
1. Flexner, W. W., "Azimuth Line of Position," pp. 475–484.
2. Birkhoff, Garrett, "What Is a Lattice?" pp. 484–487.
3. Boyer, C. B., "An Early Reference to Division by Zero," pp. 487–491.
4. Geiringer, Hilda, "The Geometrical Foundations of the Mechanics of a Rigid Body," pp. 492–502.

Bulletin of the Kansas Association of Teachers of Mathematics

February, 1943, vol. 17, no. 3.
1. Richardson, Moses, "On the Teaching of Elementary Mathematics," pp. 35–39.
2. Janes, W. C., "The Steel Square," pp. 39–44.
3. Borgmann, C. W. and Hutchinson, C. A., "Mathematics for Undergraduate Chemical Engineers," pp. 44–45.
4. "The Remedial Arithmetic Program of Manhattan [Kansas] Senior High School."
5. Bishop, Herbert H., "Mathematics before Graduation," pp. 46–48.

National Mathematics Magazine

October, 1943, vol. 18, no. 1.
1. Sanders, S. T., "Post-War Planning in Mathematics," p. 2.
2. Court, N. A., "On the Cevians of a Triangle," pp. 3–6.
3. Thébault, V., "Quadrangle Bordé de Triangles Isoscelès Semblable," pp. 7–13.
4. Bateman, H., "The Influence of Tidal Theory upon the Development of Mathematics," pp. 14–26.
5. Schaaf, William L., "Need for Studies in Mathematical Education," pp. 27–31.
6. McLenon, R. B., "Spherical Trigonometry—an Emergency Course," pp. 32–36.
7. Butchart, J. H., "Relation between the Radii of the Circumscribed and Inscribed Circles," p. 37.

School Science and Mathematics

November, 1943, vol. 43, no. 8.
1. Sleight, Norma, "Determination of Latitude and Longitude," pp. 701–708.
2. Brand, Louis, "Vectors and Their Uses," pp. 729–741.
3. Nyberg, Joseph A., "Notes from a Mathematics Classroom" (Continued), pp. 751–754.
4. Carnahan, Walter H., "A Program for the Improvement of High School Mathematics," pp. 758–765.
5. Read, Cecil B., "A Note on 'Notes from a Mathematics Classroom'," pp. 780–781.
6. Goldman, Bernard M., "A Proof of the Theorem of Pythagoras," pp. 781–782.

Scripta Mathematica

June, 1943, vol. 9, no. 2.
1. Goudge, Thomas A., "Science and Symbolic Logic," pp. 69–80.
2. Fraenkel, A. A., "Problems and Methods in Modern Mathematics: (2) Division of the Circle into a Number of Equal Parts and Other Problems," pp. 81–84.
3. Keyser, C. J., "Henri Poincaré's Style in Philosophy," p. 85.
4. Keyser, C. J., "Science and Superstition," p. 86.
5. Richards, John F. C., "A New Manuscript of a Rithomachia," pp. 87–99.
6. "Curiosa" (85); Memorizing the Logarithm of π to 20 decimal places.

Miscellaneous

1. Armstrong, G. F., "Getting Pupils to Work for Themselves," *School (Secondary Edition)* 32: 50–54, September, 1943.
2. Baldridge, C. C., "Number Activities; Practical Applications of Number Facts for Grade 2," *Grade Teacher*, 61: 32+, October, 1943.
3. De May, A. T., "Teaching Fractions," *Grade Teacher*, 61: 48+, September, 1943; October, 1943.
4. Epsy, G., "Description of Attempts to Meet the Mathematical Needs of High School Pupils," *Southern Association Quarterly*, 7: 319–331, August, 1943.
5. MacLean, M., "New Order in Grade IX Algebra," *School (Secondary Edition)* 32: 47–49, September, 1943.
6. Mallory, V. S. and Others, "Pre-induction Needs in Mathematics," *Education for Victory*, 2: 20–21, September 1, 1943.
7. Mohr, J. P., "Arithmetic Disabilities of Junior College Students," *Journal of the American Association of Collegiate Registrars*, 18: 274–280, April, 1943.
8. Rosenberg, R. R., "Drills in Preinduction Mathematics," *Business Education World*, 24: 17–18, September, 1943.
9. Swineford, F. and Holzinger, K. J., "Selected References on Statistics, the Theory of Test Construction and Factor Analysis," *School Review*, 51: 369–374, June, 1943.
10. Wilson, G. M., "Arithmetic Deficiencies," *Journal of Higher Education*, 14: 321–322, June, 1943.

NEWS NOTES

A portrait of Raymond Clare Archibald, who for 35 yr. served as professor of mathematics at Brown University, was presented to the University at a ceremony following Commencement exercises on Oct. 25, 1943.

Prof. C. Raymond Adams, chairman of the Department of Mathematics at Brown, made the presentation on behalf of a group of Prof. Archibald's friends and former students. Prof. Archibald recently retired from the Brown faculty and is now Professor Emeritus.

In making the presentation Prof. Adams declared: "Possessing a passion for accurate detail, systematic by nature and blessed with a memory that was the marvel of his friends, Professor Archibald gradually acquired a knowledge of mathematical books and their values which has scarcely been equalled. This knowledge and an untiring energy he dedicated to the upbuilding of the Mathematical Library of Brown University. From modest beginnings he has developed this essential equipment of the mathematical investigator to a point where it has no superior, in completeness and in convenience for the user.

"Of the men and women who have passed through the halls of Brown a host owe to him their appreciation of the aesthetic and functional significance of mathematics in present-day life. With a glowing enthusiasm for his subject, with a warm personal interest in his students, with a never failing patience and sympathy for those who really tried to learn, with a sense of loyalty and justice and a devotion to the scholarly life that were apparent to all who knew him, he made a deep and lasting impression on many men and women of Brown.

"Indeed, in Professor Archibald's theory of education the scholarly ideal is central, with all that the term implies in respect to industry and self-discipline on the part of student and scholar. During all his years at Brown he has himself been an exponent of the seventy hour work week with satisfaction and a half for overtime. And throughout long periods of peace and prosperity, when such ideals are not always popular, he held steadfastly to the belief that encouragement in self-discipline is a vital part of education."

Prof. Archibald is chairman of the Committee of the Council on "Mathematical Tables and Aids to Computations." He is the founder of the Mary Mellish Archibald Memorial Library at Mount Allison University in Canada, which was founded and developed during the years since 1905. This library of about 15,000 volumes is devoted to English and American poetry and drama, and is the finest place in Canada for the study of English, Scottish, and American ballads, folksongs, Indian music, Negro music, cowboy songs and eskimo songs.

The Springfield (Mass.) Daily Republican carried the following article on Saturday Oct. 9, 1943 concerning Dr. Rolland R. Smith, the president of the National Council of Teachers of Mathematics:

URGES SPRINGFIELD PIONEER IN NEW MATHEMATICS FIELD

Rolland Smith Tells School Board Needs of War, Postwar Years Demand Vitalized Mathematics in School Curricula

Springfield schools should pioneer in showing the nation how to make mathematics vital and practical, Rolland R. Smith of Classical high school faculty told the school board last night. "We have the facilities and the people—the cost involved for equipment and a coordinator would be small," he said. If this city doesn't take the lead promptly, some other community will, he stressed, explaining that the need for vitalized mathematics, demonstrated by the army and navy, will continue after the war.

Dr. Smith, the author of 15 mathematics textbooks and president of the National Council of Teachers of Mathematics, spent several weeks and many weekends in Washington, working out plans with the war department for the reorganization of the teaching of mathematics throughout the country. He was chairman and the only public school teacher on the first committee which reported on what changes are needed in the teaching of mathematics to meet army and navy needs. He was also the only public school teacher on the second committee which worked especially on the minimum requirements for the armed forces.

Mathematics, once the strongest subject in the school curriculum, fell into disrepute when the theory of training in mathematics as a discipline which carried over into other subjects was exploded. Since 1900, mathematics has suffered a slump throughout the country, although the East has made a better showing than the West and Middle West. The army and navy received a jolt at the showings made even by college students in mathematics test. One group of 4200 college freshmen who wished to enter the navy as officer candidates had 68 per cent failing in mathematic reasoning and 62 per cent failing in the whole test, Dr. Smith reported.

Dr. Smith's committees decided that the capable boys should stay on in the regular course but that the courses themselves be changed by having the applications be to the kind of things of use to the boys when they get into the services. For boys not able to profit by the regular courses, a special one-year course in mathematics was devised. The committee emphasized that not only the specialist but the average soldier must have a working knowledge of mathematics.

Classical high school has already made a start in following these suggestions, Dr. Smith reported. Much of the theory was dropped in the regular courses last year. The one-year course initiated this year has attracted many girls as well as boys of 16 and 17 years who are beginning to realize how handicapped they are by the lack of mathematics, Dr. Smith declared.

Asked by a board member what his definite recommendation to the school board was, Dr. Smith said that he preferred not to make one

at this time. He wished the board to take time to consider the matter before deciding whether or not this city should take the lead in the new teaching of the mathematics. The changes should affect every grade, from the kindergarten up, he suggested.

Dr. H. C. Christofferson, former president of the National Council of Teachers of Mathematics, addressed the Ohio teachers of mathematics at Lima, Ohio, on October 29, 1943, on the topic, "Mathematics Which Functions in War and Peace."

The following program was given at the meeting of the Northeast District of the Oklahoma Educational Association at Tulsa on October 29, 1943:

Chairman:—J. E. Sullivan, Muskogee
Secretary:—Nelle Burton, Tulsa

Music, Orchestra, Central High School, Tulsa, Clarence F. Gates, Director
"The Urgency of Teaching Mathematics in High School," Major Leo Towers, U. S. Engineers, Tulsa
"Notes on Basic Preparation," W. S. Bishop, Northeastern State College, Tahlequah
"Mathematics in War and Peace," Dr. J. O. Hassler, Oklahoma University, Norman
Business Meeting—Officers elected: President—Eunice Lewis, Tulsa; Vice-President—Naomi Williams, Skiatook; Secretary—Mrs. H. T. Fegan

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The mathematics section of the Western Pennsylvania State Education Association held an exhibit of mathematical instruments, figures, models and related material made by high school, private school and college students in Pittsburgh, Pennsylvania, October 22 and 23. A wide variety of the many phases of mathematics was shown in the exhibit including a number of commercial contributions.

Miss Mary A. Potter, former president of the National Council of Mathematics Teachers and present supervisor of mathematics in Racine, Wisconsin, delivered a timely address, "The Armed Forces Draft Mathematics."

The following teachers in the Pittsburgh area served on committees: exhibit, Amelia Richardson, Catherine Lyons, Ida Price, Dr. Edwin G. Olds, Dr. James S. Taylor, E. J. Owens, Mercedes Schramm, and V. C. Veverka; publicity, Clementina George, Joseph Lynch, Anne Rightmire, Dr. W. J. Wagner, Bertha Kirkpatrick, Patrick Cronin, and Frank B. Ankeney.

Many other teachers gave their wholehearted support in making the affair a success.—Clementina George, Chairman Publicity Committee.

The Alabama Branch of the National Council of Teachers of Mathematics met in Birmingham on March 25, 1943. The officers were:

President—Lester M. Garrison, Auburn
Vice-President—Robert M. Snuggs, Jr., Montgomery
Secretary-Treasurer—Miss Jeannette Garrett, Birmingham

The following program was given:

10:15 Business Meeting
Greetings from the National Council, J. Eli Allen
10:30 Address—*High School Mathematics in War and Peace*, Miss Margaret McPherson, Woodlawn High School, Birmingham
11:10 Address—*The Mathematics Program at the Army Air Forces Pre-Flight School (Pilot)*, Lt. Neal B. Andregg, Head of Department of Mathematics, Army Air Forces Pre-Flight School (Pilot), Maxwell Field

New officers elected for the next were: Dr. Rosa Lee Jackson, Alabama College, president; M. H. Pearson, Montgomery, vice-president; Mrs. Rosalee Blackwell, Ensley, secretary-treasurer.

The first meeting of the Men's Mathematics Club of Chicago and the Metropolitan area was held on October 15, 1943, at the Central Y.M.-C.A. Mr. Marx Holt, Principal of Fiske School, spoke on "Post-War Magic."

H. C. Torreyson
Chairman Program Committee

The fall meeting of the Mathematics Section (Eastern Division) of the Colorado Eduation Association was held on October 29, 1943, at Denver. Professor J. O. Hassler gave an address on "Teaching Mathematics Effectively for War or Peace," and Lt. Col. G. Everett Hill, Jr., spoke on "Mathematics for the War Effort." Both of these speeches were given at a joint session of the above Division and the Rocky Mountain Section of the Mathematical Association of America. The program of the Eastern Division itself follows:

1. Brief business meeting
2. "The New Arithmetic Program in the Elementary School," Anna E. Kane, Denver
3. "The Scope and Sequence of Junior High School Mathematics," Ernest R. Bails, Denver
4. "Constructions with a Marked Rule," Claribel Kendall, Boulder

MEMBERS OF THE EXECUTIVE COUNCIL

President—Wendell Wolf, 2821 W. 34th Avenue, Denver
Vice-President—A. E. Mallory, C.S.C.E., Greeley
Secretary-Treasurer—Mary Doremus, 1416 Penn, Denver

Term expires 1943: Ernest Cruse, Greeley; Dwight Gunder, Ft. Collins; Margaret McGinley, Denver; Frances Smith, Sterling.

Term expires 1944: Margaret Aylard, Denver; Dan Beattie, Ft. Collins; Arthur Lewis, Denver; Isabelle Staub, Denver.

Term expires 1945: R. T. Ashbaugh, Longmont; Katherine Compton, Denver; Kenneth Gorsline, Denver; Ruth Irene Hoffman, Denver.

The Association of Mathematics Teachers of New Jersey had for its main topic "Understanding Our World Through Mathematics" at the Hotel Pennsylvania on November 12, 1943.

MORNING PROGRAM

"Understanding Our World Through the Mathematics of Geography and Mapping," Miss Mildred A. Fink, Roosevelt Junior High School, Westfield, N. J.

"Understanding Our World Through the Practical Mathematics of the Senior High School," Mr. Ferdinand Kertes, Head of the Department of Mathematics, Perth Amboy High School, Perth Amboy, N. J.
"New Horizons," Dr. E. H. C. Hildebrandt.

AFTERNOON PROGRAM

"Present Day Applications of Mathematics," Dr. J. W. Barker, Dean—School of Engineering, Columbia University, now Special Assistant to the Secretary of the Navy.
"The Army's Minimum Needs in Arithmetic," Dr. W. A. Brownell, Professor of Mathematics, Duke University, now Consultant in War Department.

SCHOOLS-AT-WAR REPORT TO OUR ALLIES

From now on it won't be just the outstanding leaders of two great allied nations who will help cement the friendly relationship of the United States and Great Britain. The school children of America are entering the picture as young diplomats in their own right.

In order that the boys and girls of England may get better acquainted with their compatriots across the sea, creating an even closer bond of understanding, 61 Schools-at-War scrapbooks, made by the school children of America, are now ready for embarkation. They were submitted to the War Finance Division of the Treasury Department and are being shipped at the request of the British Division of the Office of War Information.

Upon arrival overseas, these graphic portrayals of what American boys and girls are doing in wartime will be displayed first at the U. S. Information Library of the American Embassy in London. From there they will be sent on a "Good-Will Tour" throughout the country—to be studied by British educational groups, including not only individual schools but teachers associations.

These scrapbooks were originally made by schools participating in the War Savings program, in order to report their wartime activities. Soon they became all embracing—including student compositions, art work, photographs, poems, letters, plans of Red Cross activities, scrap drives, victory gardening, plane spotting, and food conservation, as well as Stamp sale devices.

Last year, some 28,000 schools enrolled as Schools-at-War. After local and regional exhibits, their scrapbooks were submitted to state War Finance Offices. Five scrapbooks from each state were then selected for the national Schools-at-War Exhibit, held in Indianapolis, Indiana, during the annual meeting of the National Education Association, June 25 to 30.

Several months ago Washington officials sent four scrapbooks as advance emissaries to the British. So enthusiastically were they received, depicting as they did the vivid story of what American boys and girls are really like, what they feel they are fighting for, and what they are doing toward victory, that a call came over the Atlantic for more.

The scrapbooks are from all sections of the country—from Maine to Florida to Idaho. They should prove quite exciting to the British children—seeing palms about tropical school buildings, snow about others, and small adobe boxes set in flat deserts. The contrasts will be startling, denoting the vastness of our country.

They'll range from a scrapbook representing a city system of 52,806 pupils right on down to a one-room school with only four pupils, all of whom are buying bonds. The one from the Virginia School for the Blind should attract attention because of the Braille used in the labels.

This fascinating panorama of American Schools-at-War will be exhibited in England to coincide with the visit of Worth McClure, President of the American Association of School Administrators, who is being sent abroad to speak to British educators and thus help to interpret the United States and our educational system. Among the first four books sent over was that from the Seattle public schools, of which Mr. McClure is superintendent.

The following notes may be of interest to our readers:

Philadelphia

Toward the end of the Third War Loan Drive, Philadelphia school children organized a "commando attack" on the shortage the city faced in its Bond quota. After a meeting of school principals, it was decided to enroll 100,000 public-school pupils in Philadelphia's Blue Star Brigade of volunteer solicitors. In response to requests from both public and parochial schools, the last week of the campaign was designated "Buy a Bomber Week," so that pupils might credit their sales to the schools' individual drives to sell enough Bonds to buy a bomber.

Detroit

During the Third War Loan the public schools of Detroit assumed a War Savings quota of $10 million, one-tenth more than their total sales for 1942–43. To facilitate the sale of Bonds to teachers, students, and parents, 150 schools became sub-issuing agents for the sale of Bonds. This was arranged through the business office of the Detroit Board of Education and in conjunction with the Federal Reserve Bank.

Endicott, N. Y.

Students at Endicott High School, N. Y., are receiving big dividends on their War Savings investments. They have learned that the *Endicott Special*, one of three fighter planes financed by student purchases of Bonds and Stamps, bagged three Japanese bombers and one fighter on its maiden flight over New Guinea. The campaign to name a plane was initiated by the high-school paper and attracted support of the eight other schools in the system. Endicott boys and girls spurred by the news of "their" plane, have nearly completed the "purchase" of a fourth plane with money earned during the summer vacation.